ISSN 0029 - 3865

CBPF-NF-067/85

PHASE-SPACE DYNAMICS OF BIANCHI IX COSMOLOGICAL MODELS*

by

I. Damião Soares

Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

*Invited paper to the Marcel Grossmann Volume, 1985, ed. R. Ruffini. This paper reports part of a program in collaboration with J.R.T. de Mello Neto and J. Koiller.

We discuss here the complex phase-space dynamical behaviour of a class of Bianchi IX cosmological models, as the chaotic gravitational collapse due to Poincaré's homoclinic phenomena, and the n-furcation of periodic orbits and tori in the phase space of the models. Poincaré maps which show this behaviour are constructed numerically and applications are discussed.

Key-words: Bianchi IX models; Chaotic gravitational collapse; Bifurcation of orbits.

In this paper we intend to discuss the complex dynamical behaviour of a class of Bianchi IX models. In the dynamics of these models we observe not only the possibility of a chaotic gravitational collapse, but also phenomena as n-furcation of the orbits, the presence of stochastic regions and stability islands in the phase plane of the system. Stochastic properties in the dynamics of a Bianchi IX cosmological model were first discussed by Belinskii, Khalatnikov and Lifshitz [1], on examining the approach to the singularity of a general Bianchi IX solution. Later Barrow and Chernoff [2] derived some maps for the dynamics of the mixmaster universe [3] which also exhibit strong stochastic properties.

The class of models considered here have the topology $R\times S^3$. Here S^3 is Hopf's fiber bundle with base space S^2 and fiber homeomorphic to S^1 [4]. The temporal coordinate is defined on R and the geometry is given by

$$ds^2 = dt^2 - (A^2(t)\, g_V + B^2(t)\, g_H) \tag{1}$$

where g_V is the geometry of the fiber S^1 and g_H is pulled back from the geometry of the base space S^2. The radius of the 2-sphere S^2 and the radius of S^1 are time-dependent, with respective time dependence B(t) and A(t), and their dynamics is given by Einstein equations with the cosmological constant term. The matter content of the model is a perfect fluid with matter-energy density ρ, pressure p and four-velocity $\partial/\partial t$. Einstein equations for (1) reduce to three independent equations. Two of them define ρ and p, and the third one yields the differential equation

$$\frac{\ddot{B}}{B} + \left(\frac{\dot{B}}{B}\right)^2 - \frac{\ddot{A}}{A} - \frac{\dot{A}\dot{B}}{AB} + \frac{1}{B^2} - \frac{A^2}{B^4} = 0 \tag{2}$$

The physically admissible solutions of (2) must be restricted by the energy conditions [5] that ρ and p must satisfy. In all cases discussed in this paper the energy conditions are satisfied.

From (2) we examine the following possibilities:

(I) Oscillations of the sector S^2 of the geometry: $A^2 = \lambda^2$, and the dynamics of B(t) is described by the hamiltonian $H = \frac{1}{2}(\dot{q})^2 + V(q) = C$, with $V(q) = 2q - 2\lambda^2 \ln q$ and C = const, where we denoted $q = B^2(t)$. The potential V(q) has one absolute minimum for $q = \lambda^2$ and corresponds to the configuration of the Einstein Universe. The trajecto-

ries of the system in the phase plane $(q, \dot{q})$ are closed curves about $(\lambda^2, 0)$, whose period depends on the parameter $\varepsilon^2 = C - 2\lambda^2(1 - \ln\lambda^2)$. They can be confined to any neighborhood of the stability point $(\lambda^2, 0)$.

(II) Oscillations of the sector S^1: $B^2 = \lambda^2$, and the dynamics of $A(t)$ is given by the Hamiltonian $H = \frac{1}{2}(\dot{A})^2 + V(A) = D$, where $V(A) = \frac{1}{4\lambda^4}(A^4 - 2\lambda^2 A^2)$ and D is a constant. The minimum of the potential also corresponds to the configuration of the Einstein universe. The points $A = 0$ are physical singularities of the model. We remark that for the value $D = 0$ the trajectories in the phase plane $(A, \dot{A})$ are homoclinic curves [6] which link the homoclinic point $(0, 0)$ to itself.

(III) Gravitational interaction of the sectors S^1 *and* S^2: we consider the special mode in which the oscillations in the sector S^2 excite the degree of freedom of S^1, via gravitational interaction. For this we take $B(t, \varepsilon)$ a periodic solution of *(II)* and substitute into (2) to obtain

$$A'' + \frac{T^2(\varepsilon)}{B^2}(A^3 - \lambda^2 A) = 0 \tag{3}$$

Here a prime denotes $d/d\eta$ where the variable η is defined by $d\eta = (T(\varepsilon)\, B)^{-1}\, dt$. We note that the period $T(\varepsilon)$ of the function $B(t, \varepsilon)$ is normalized to 1 in the variable η. Equation (3) properly describes the excitation of the degree of freedom of S^1 by oscillations in S^2: in fact $(A^2 = \lambda^2, A' = 0)$ is a solution of (3) (corresponding to mode *(I)*), and we can show by linearizing (3) about $A^2 = \lambda^2$ that any small fluctuation $u = A - \lambda$ is highly unstable and grows rapidly into the non-linear regime due to the oscillations of the sector S^2 [7].

The system (3) has a complex dynamical behaviour as we proceed to discuss, and we distinguish two set of phenomena. In what follows we take $\lambda^2 = 1$.

First let us consider the homoclinic curves $D = 0$ of case *(II)* which smoothly link the unstable fixed point $(0, 0)$ to itself. The introduction of a small perturbation according to (3) by infinitesimal oscillations of the sector S^2 (namely for ε^2 infinitesimal in mode *(I)*) is sufficient to break this smooth link and produce the homoclinic phenomena of Poincaré [8] in a small neighborhood Γ of the homoclinic curves $D = 0$. The homoclinic phenomena are the basis of the chaotic behaviour of the model, and we can always program the dynamics (by properly choosing initial conditions in a certain subet of Γ), so that for *any* positive integer n ($n = \infty$ included) the universe undergoes n non-periodic oscillations (each oscillation requiring a long time) before collapsing (for $n = \infty$ the universe undergoes periodic oscillations) [9].

Second, let us consider (3) for the case $B^2 = 1$ (corresponding to mode *(II)*) and restrict ourselves to the dynamical region inside the separatrix $D = 0$ (namely $-1/4 < D < 0$). As well known, the phase space of this unperturbed system is foliated by 2-dim invariant tori [6,10,11] each one characterized by its frequency $\omega(D)$. Introducing in (3) the infinitesimal perturbation $B^2 = 1 + \varepsilon \cos 2\pi\eta$, the KAM theorem [12] tells us that most of the tori are preserved by the perturbation, namely those whose unperturbed frequency is a diophantine irrational. These preserved tori enclose destroyed regions (corresponding to unperturbed tori with rational frequency and their neighborhood) whose Lesbegue measure goes to zero as $\varepsilon \to 0$. The associated Poincaré map [6,11] to period 1 exhibit — for these destroyed regions — a structure of elliptic and hyperbolic fixed points [10, 13] which correspond to periodic orbits of the system. The neighborhood of each hyperbolic point has chaotic dynamics, and the neighborhood of each elliptic point reproduces again the same picture — diophantine irrational tori enclosing destroyed regions whose Poincaré map has a structure of elliptic and hyperbolic fixed points, corresponding to period orbits of larger periods, etc. — to arbitrarily small scales [14].

For large ε, the KAM no longer applies but we may examine the dynamics of (3) by constructing the associated Poincaré map to period 1 numerically. We have done this for a large set of initial conditions and a large range of the parameter ε [15]. In the graphs (X, Y) stand for (A, A').

For large ε the structure of irrational tori is still maintained in a region close to the stability point (1, 0). Fig. 1 shows the torus section obtained by numerically constructing the Poincaré map for initial conditions ($A = 0.96$, $A' = 0.00$) and $\varepsilon = 2.0$

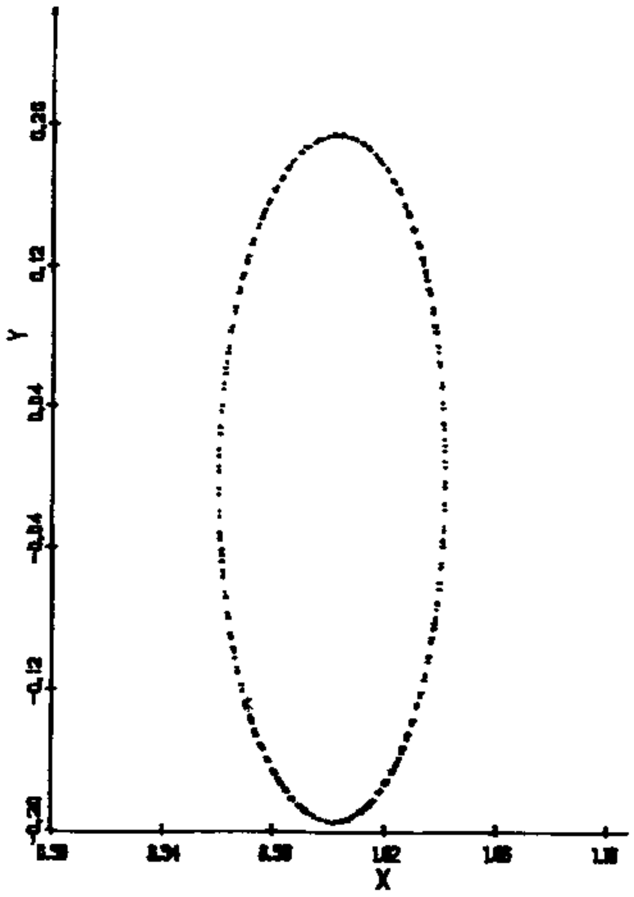

FIGURE 1

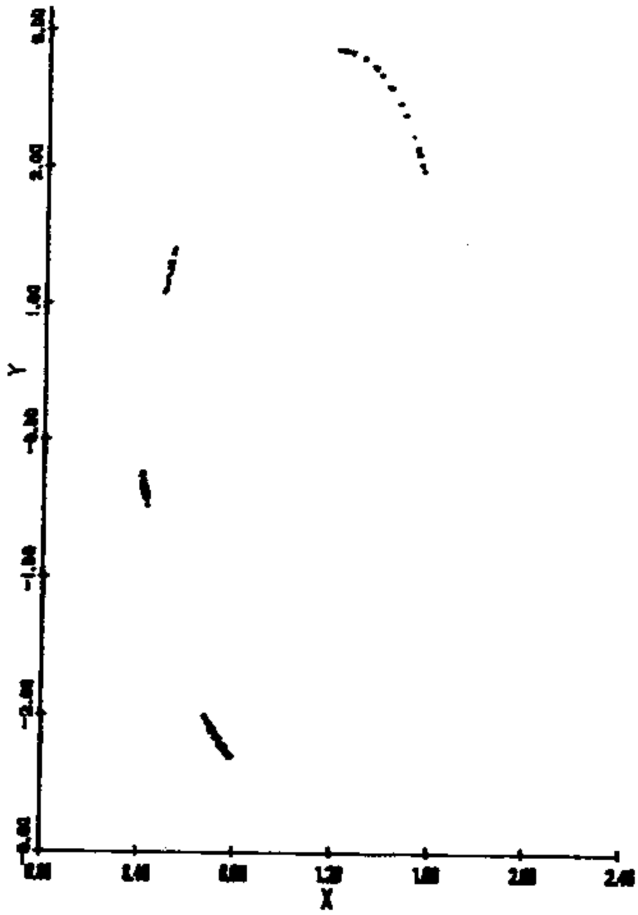

FIGURE 2

Some of the tori n-furcate as we vary the parameter ε to a certain value. In Fig. 2 we show the Poincaré map corresponding to a 4-furcated torus for A = 0.502, A' = 1.22 and ε = 1.0. Note that a T=4 periodic orbit is enclosed by the torus.

The chaotic region which appers about an infinitesimal neighborhood of the unperturbed separatix D=0 — due to Poincaré's homoclinic phenomena discussed above for ε^2 infinitesimal — tends to increase for increasing ε and to occupy a large area of the phase plane of the system, as shown by the Poincaré map in Fig. 3 for A = 0.55, A' = 0.007 and ε = 2.828 (three hundred points plotted). However inside this chaotic region we still find islands of stability as shown in Fig. 4 by the Poincaré map of a 5-furcated torus, enclosing a T=5 periodic orbit (for initial conditions A = 0.5, A' = - 0.00001 and ε = 2.828).

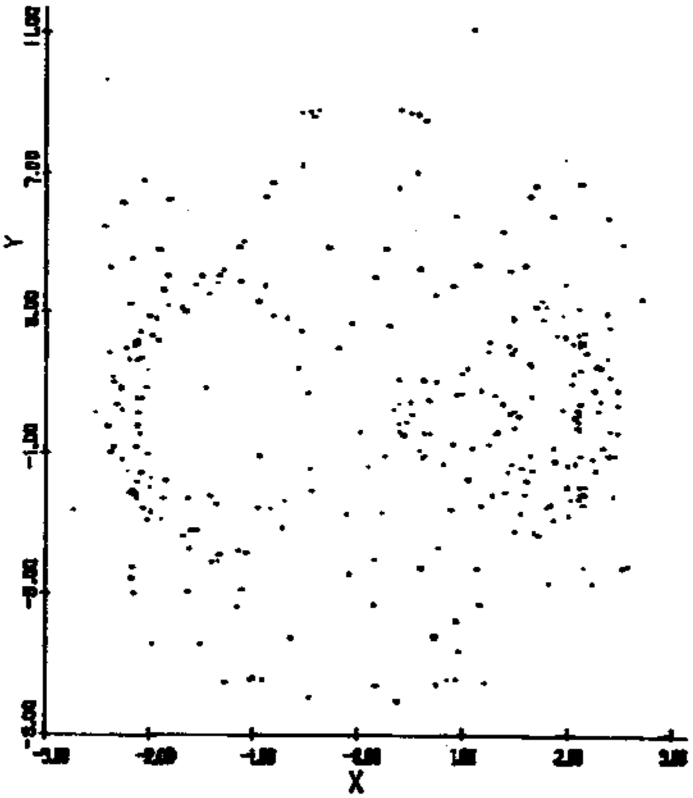

FIGURE 3

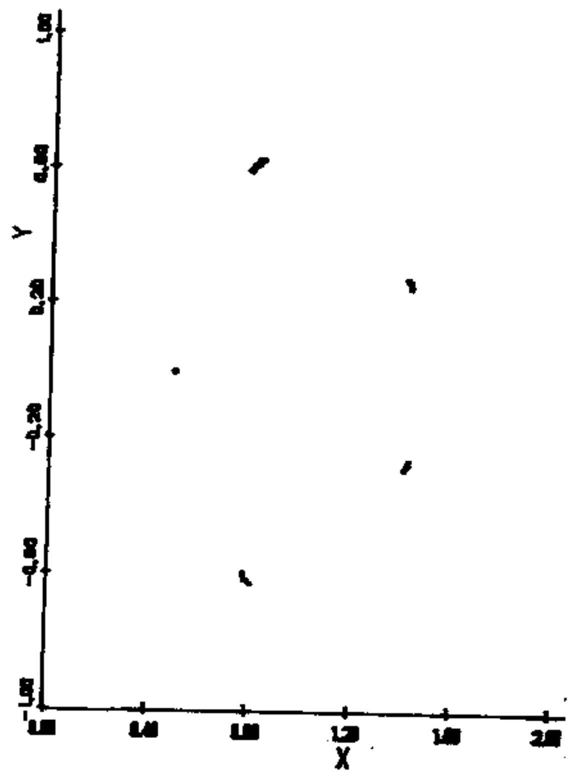

FIGURE 4

Some possible physical applications can be discussed. Small fluctuations in matter-energy density ρ, pressure p, etc. over this background geometry can be made to grow (for specific wave lenghts) by a res onance phenomena when a bifurcation of orbits occurs. Let us take a T=1 periodic soution $A(\eta, \varepsilon)$ which — by adiabatic variation of ε — n-furcates to a T=n periodic solution. By a resonance phenomenon in the linear equations governing the fluctuations, matter fluctuations can be amplified whose wave lenght is equal to $2\pi n$ and thus create a selected spectrum of perturbations in the matter fluctuations. This work is now in progress.

Final Note: for computational simplicity we have taken for $B^2(\eta, \varepsilon)$ the approximate expression

$$B^2(\eta,\varepsilon) = 1 + \varepsilon \cos 2\pi\eta + \varepsilon^2 \left[\frac{1}{2} - \frac{1}{2}\sin^2 2\pi\eta - \frac{1}{6}\cos 4\pi\eta\right].$$

Larger dots in the graphs represent actually several near points.

REFERENCES

1) V.A. Belinskii, I.M. Khalatnikov and E.M. Lifshitz, Adv. Phys. 19 (1970) 525; 31 (1982) 639; J. Stat. Phys. 38 (1985) 97.

2) D.F. Chernoff and J.D. Barrow, Phys. Rev. Lett. 50 (1983) 134.

3) C.W. Misner, Phys. Rev. Lett. 22 (1969) 1071.

4) N. Steenrod, The topology of fiber bundles (Princenton Univ. Press, Princenton, 1951).

5) S.W. Hawking and G.F.R.Ellis, The large scale structure of space-time (Cambridge Univ. Press, London, 1973).

6) J. Guckenheimer and P.J. Holmes, Nonlinear oscillations, dynamical systems and bifurcations of vector fields (Springer, Berlin, 1983).

7) J.R.T. de Mello Neto and I. Damião Soares, to be published.

8) H. Poincaré, Les méthodes nouvelles de la mécanique celeste, vol. 3 (Gauthier-Villars, Paris, 1899); J. Moser, Stable and random motions in dynamical systems (Princeton Univ. Press, Princeton, 1973).

9) J. Koiller, J.R.T. de Mello Neto and I. Damião Soares, Phys. Lett. 110A (1985) 260.

10) V.I. Arnold, Mathematical methods of classical mechanics (Springer, Berlin, 1978).

11) V.I. Arnold and A. Avez, Ergodic problems in classical mechanics (Benjamin, New York, 1968).

12) V.I. Arnold, Sov. Math. Dokl. 3 (1963) 136; J. Moser, Nachr. Akad. Wiss Göttingen. Math. Phys. Kl (1962) 1.

13) G.D. Birkhoff, Nouvelles recherches sur les systemes dynamiques, Mem. Pont. Acad. Novi Lyncaci (1935) ch. 4.

14) M.J. Berry, Regular and irregular motion, in: Topics in nonlinear dynamics, ed. S. Jorna (AIP Conf. Proc., New York, 1978) p.16.

15) J.R.T. de Mello Neto and I. Damião Soares, to be published.