ISSN 0029 - 3865

CBPF-NF-064/85

ON THE DIFFERENTIAL EQUATION OF THE COSMIC RAY'S NUCLEONIC COMPONENT WITH DISTRIBUTION OF ELASTICITY

by

F.M. de Oliveira Castro

Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

Abstract

The differential equation of cosmic ray's nucleonic component is integrated with distribution of elasticity.

Key-words: Cosmic rays; Nucleonic components; Elasticity.

The differential energy spectrum of the primary cosmic ray nucleons at the atmosphere depth $x(g/cm^2)$ may be obtained integrating the differential equation

$$\lambda \frac{\partial F(x,E)}{\partial x} = -F(x,E) + \int_0^1 F(x, \frac{E}{\eta}) f(\eta) \frac{d\eta}{\eta} \tag{1}$$

with the initial condition

$$F(0,E) = G(E) \quad . \tag{2}$$

The function $G(E)$ that represents the primary spectrum of protons at the top of the atmosphere is supposed to be non negative continuous and bounded in the interval $0 < a \leqq E < \infty$. The existence of the integral $\int_E^\infty G(E)dE$, for $E \geqq a$, must also be stated because it represents the primary integral spectrum. The function $f(\eta)$ represents the distribution of elasticity of the collisions of the nucleons with the air nuclei and λ is the mean interaction length of the collisions. As it is well known, this equation can be easily integrated using the method of Mellin's transforms, but, so doing we obtain the real solution represented by a contour integral in the complex domain and, only in very few particular cases this integral can be evaluated exactly and we must use some approximate method for estimate it as for example the saddle point method.

The integration of the equation (1) which takes in to account the distribution of elasticity is only a little more complicated than the integration of the simplified diffusion equation

$$\frac{\partial F(x,E)}{\partial x} = -\frac{1}{\lambda} F(x,E) + \frac{1}{\lambda(1-k)} F(x, \frac{E}{1-k}) \tag{3}$$

with the initial condition

$$F(0,E) = G(E) \tag{4}$$

where k represents the mean inelasticity of the collisions.

As it was shewn by the author in a previous paper[1], the integration of the equation (3) can be easily performed using the method of successive approximations. This is the reason that we shall use the same method to treat equation (1).

To simplify the work of performing the successive approximations first we put

$$F(x,E) = e^{-x/\lambda}\, y(x,E) \tag{5}$$

In so doing equation (1) and the respective initial condition, become

$$\frac{\partial y(x,E)}{\partial x} = \frac{1}{\lambda}\int_0^1 y(x,\frac{E}{\eta}) f(\eta)\frac{d\eta}{\eta} \tag{6}$$

$$y(0,E) = G(E) \quad . \tag{7}$$

Now we make the following successive approximations

$$\begin{cases} y_0(x,E) = G(E) \\ y_n(x,E) = G(E) + \frac{1}{\lambda}\int_0^x dx \int_0^1 y_{n-1}(x,\frac{E}{\eta}) f(\eta)\,\frac{d\eta}{\eta} \quad . \end{cases} \tag{8}$$

---

[1]Oliveira Castro, F.M., (1977) An. Acad. Brasil. Ciênc. 49 - 113 - 118.

So doing we obtain successively

$$y_1(x,E) = G(E) + \frac{x}{\lambda}\int_0^1 G(\frac{E}{\eta_1}) f(\eta_1)\frac{d\eta_1}{\eta_1}$$

$$y_2(x,E) = G(E) + \frac{x}{\lambda}\int_0^1 G(\frac{E}{\eta_1}) f(\eta_1)\frac{d\eta_1}{\eta_1} + \frac{x^2}{2!\lambda^2}\int_0^1\int_0^1 G(\frac{E}{\eta_1\eta_2}) \times$$

$$\times \frac{f(\eta_1)}{\eta_1}\ \frac{f(\eta_2)}{\eta_2}\ d\eta_1 d\eta_2$$

$$y_n(x,E) = \sum_{\nu=1}^{n} \frac{(x/\lambda)^\nu}{\nu!}\int_0^1\cdots\int_0^1 G(\frac{E}{\eta_1\cdots\eta_\nu})\frac{f(\eta_1)}{\eta_1}\cdots\frac{f(\eta_\nu)}{\eta_\nu}\ d\eta_1\ldots d\eta_\nu$$

$$+ G(E)$$

a) Convergence of the Succession $y_n(x,E)$

Note that if $G(E)$ is non negative and bounded in the interval $I = [a,\infty)$, $a > 0$, we have $G(E) \leqq M$ for $E \varepsilon I$, where M is some positive constant. The function $f(\eta)$ which is a datum of our problem and is essentially positive. We assume the existence of the integral $\int_0^1 f(\eta)d\eta/\eta = \beta$.

Now consider the series $S = \sum_{n=0}^{\infty} u_n(x,E)$ whose n th partial sum S_n is $y_n(x,E)$. It is a series of positive terms. S_n is bounded in any set (T) such that $0 \leqq x \leqq X$; $a \leqq E \leqq b$, $a > 0$, because

$$S_n = y_n(x,E) \leqq M\sum_{\nu=0}^{n} (\frac{x/\lambda}{\nu!})^\nu\ \beta^\nu < M\,e^{\beta x/\lambda} < M\,e^{\beta X/\lambda} \quad .$$

The uniform convergence of the exponential in the set (T) assures the uniform convergence of the series S to a function $y(x,E)$, on (T)

The convergence is uniform relatively to both x and E, on (T). Then we have

$$y(x,E) = \sum_{\nu=1}^{\infty} \frac{(x/\lambda)^{\nu}}{\nu!} \int_0^1 \cdots\cdots \int_0^1 G\left(\frac{E}{\eta_1\cdots\eta_\nu}\right) \frac{f(\eta_1)}{\eta_1} \cdots \frac{f(\eta_\nu)}{\eta_\nu} \, d\eta_1\cdots d\eta_\nu \tag{9}$$

$$+ G(E)$$

b) Synthesis of the Solution

We must proof that the sum $y(x,E) = \lim_{n\to\infty} y_n(x,E)$, in (T) is effectively a solution of the equation (6) with the initial condition (7).

In fact, we have

$$y(x,E) = \lim_{n\to\infty} y_n(x,E) = G(E) + \lim_{n\to\infty} \frac{1}{\lambda} \int_0^x dx \int_0^1 y_n(x,\tfrac{E}{\eta}) f(\eta) \frac{d\eta}{\eta}$$

$$= G(E) + \frac{1}{\eta} \int_0^x dx \lim_{n\to\infty} \int_0^1 y_n(x, \tfrac{E}{\eta}) f(\eta) \frac{d\eta}{\eta} \quad .$$

Since $y_n(x,E) \to y(x,E)$ uniformly in (T) we can take the limit under the sign of integration. Thus we have

$$y(x,E) = G(E) + \frac{1}{\lambda} \int_0^x dx \int_0^1 y(x,\tfrac{E}{\eta}) f(\eta) \frac{d\eta}{\eta} \quad .$$

Hence $y(x,E)$ is a solution of equation (6).

Note that $y(x,E)$ is continuous in any set (T): $0 \leq x \leq X$; $a \leq E \leq b$, $a > 0$, X and b arbitrary, but finite.Moreover, for < x fixed y(x,E) is bounded in the intervals $0 < \alpha \leqq E < \infty$.

c) Uniqueness of the Solution

Suppose that there are two solutions y(x,E) and z(x,E) of the equation (6) both continuous on every rectangle

$$T: [0 \leqq x \leqq X,\ a \leqq E \leqq b] \quad , \quad a > 0 \ , \ b > a \ , \ X > 0.$$

Then the difference u(x,E) = y(x,E)-z(x,E) must satisfy the homogeneous equation

$$u(x,y) = \frac{1}{\lambda}\int_0^x dx \int_0^1 u(x,\frac{E}{\eta}) f(\eta)\frac{d\eta}{\eta} \quad . \tag{10}$$

Now substituting iteratively in (10) the function u(x,E) under the sign of integration by its own value given by (10) we have successively

$$u(x,E) = \frac{1}{\lambda}\int_0^x dt_1 \int_0^1 f(\eta_1)\,\frac{d\eta_1}{\eta_1}\,u(t_1,\frac{E}{\eta_1})$$

$$= \frac{1}{\lambda}\int_0^x dt_1 \int_0^1 \frac{f(\eta_1)}{\eta_1}\,d\eta_1 \left[\frac{1}{\lambda}\int_0^{t_1} dt_2 \int_0^1 f(\eta_2)\frac{d\eta_2}{\eta_2}\,u(t_2,\frac{E}{\eta_1\eta_2}) = \right.$$

$$= \frac{1}{\lambda^2}\int_0^x dt_1 \int_0^{t_1} dt_2 \cdot \int_0^1 \frac{f(\eta_1)d\eta_1}{\eta_1}\int_0^1 \frac{f(\eta_2)d\eta_2}{\eta_2}\;u(t_2,\frac{E}{\eta_1\eta_2}) =$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \frac{1}{\lambda^n}\int_0^x dt_1 \ldots \int_0^{t_{n-1}} dt_n \cdot \int_0^1 \frac{f(\eta_1)d\eta_1}{\eta_1}\ldots\int_0^1 \frac{f(\eta_n)d\eta_n}{\eta_n}\times u(t_n,\frac{E}{\eta_1\ldots\eta_n})$$

Since $u(t,E)$ is the difference of two functions continuous and bounded in the interval $a \leqq E < \infty$; $a > 0$, we can write $\left|u(t_n, \frac{E}{\eta_1 \ldots \eta_n}\right| \leqq M_1$ where M_1 is some positive constant. Thus we have, for every fixed $\underline{x}$:

$$|u(x,E| \leqq \frac{x^n}{n!} \times \beta^n \qquad \text{where} \qquad \beta = \int_0^1 \frac{f(\eta)d\eta}{\eta} = > 0$$

So that $u(x,E) \to 0$, when $n \to \infty$, for any x fixed such that $0 \leqq x < \infty$. Therefore $y(t,E) = z(x,E)$. The proof is complete.

Finally, taking equations (5) and (9) into account we have

$$F(x,E) = e^{-x/\lambda} \sum_{\nu=1}^{\infty} \frac{(x/\lambda)^{\nu}}{\nu!} \int_0^1 \ldots \int_0^1 G\left(\frac{E}{n_1 \ldots \eta_\nu}\right) \frac{f(\eta_1)}{n_1} \ldots$$

$$\ldots \frac{f(\eta_\nu)}{n_\nu} \, d\eta_1 \ldots d\eta_\nu + G(E)$$

which is the desired solution of eq. (1), with the initial condition $F(0,E) = G(E)$.