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WEYL (INTEGRABLE) SPACETIMES AS A MODEL FOR OUR COSMOS*

by

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ABSTRACT:

We present a discussion on Weyl (integrable) spacetimes as a viable model for our cosmos, and on the physical implications for systems non-minimally coupled to gravity.

Key-words: Weyl space-times; Non-minimal coupling; Cosmological constant.

1. INTRODUCTION

In the spirit of the SILARG, I shall present here some results recently obtained by the Group of Cosmology and Gravitation of the CBPF, concerning the possibility that Weyl integrable spacetimes could furnish a physically viable model for our cosmos. In order to illustrate the physical implications of this approach we investigate the case of electromagnetism non-minimally coupled to gravity and the problem of generating a cosmological constant.

2. WEYL SPACETIMES

A Weyl spacetime (WST)[1] is an affine manifold endowed, besides the metric $g_{\mu\nu}$, with a four-vector $\omega_\mu(x)$, the so-called gauge vector, such that the length ℓ of a given vector $V_\mu(x)$, defined by $\ell^2 = g_{\mu\nu}V^\mu V^\nu$, changes under an infinitesimal transplantantion of V_μ according to $d\ell = \ell\, \omega^\mu dx_\mu$, that is, while the metric provides a (local) definition of the length of V_μ, the gauge vector furnishes the (local) variation of this length.

The affine (symmetric) connections $\Gamma^\alpha_{\mu\nu}(x)$ of this manifold are given by[2]

$$- \Gamma^\alpha_{\mu\nu} = \{^{\alpha}_{\mu\nu}\} - \frac{1}{2} g^{\alpha\lambda} \left[g_{\lambda\mu}\omega_\nu + g_{\lambda\nu}\omega_\mu - g_{\mu\nu}\omega_\lambda \right] , \tag{1}$$

where $\{^{\alpha}_{\mu\nu}\}$ are the usual Christoffel symbols of General Relativity, built exclusively with $g_{\mu\nu}$ and its derivatives, so that the covariant derivative $g_{\mu\nu;\alpha}$ of the metric turns out to be

$$g_{\mu\nu;\alpha} \equiv g_{\mu\nu,\alpha} - \Gamma^\lambda_{\mu\alpha}\, g_{\lambda\nu} - \Gamma^\lambda_{\nu\alpha}\, g_{\lambda\mu} = g_{\mu\nu}\omega_\alpha \quad . \tag{2}$$

Therefore, when the gauge vector is null, one gets $\Gamma^\alpha_{\mu\nu} = \{^{\alpha}_{\mu\nu}\}$ and so

$$g_{\mu\nu;\alpha} = g_{\mu\nu,\alpha} - \{^{\lambda}_{\mu\alpha}\}\, g_{\lambda\nu} - \{^{\lambda}_{\nu\alpha}\} g_{\lambda\mu} \equiv g_{\mu\nu||\alpha} = 0 \ , \tag{3}$$

which charactherizes the structure of a Riemann spacetime (RST). Thus, the usual RST of General Relativity is a particular case of the WST, occurring when $\omega_\mu = 0$. Other relevant geometric quantities are obtained using the prescriptions of affine geometry; for example, Ricci's tensor $R_{\alpha\beta}$ is

$$R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta} = \overline{R}_{\alpha\beta} + \frac{3}{2} w_{\alpha||\beta} - \frac{1}{2} w_{\beta||\alpha} + \frac{1}{2} w_{\alpha} w_{\beta} + \frac{1}{2} g_{\alpha\beta}\left[w^{\mu}{}_{||\mu} - w^{2}\right] \quad (4)$$

and the scalar of curvature R is

$$R = g^{\alpha\beta} R_{\alpha\beta} = \overline{R} + 3\,(w^{\mu}{}_{||\mu} - \frac{1}{2} w^{2})\ , \quad (5)$$

so that Einstein's tensor in a WST is

$$G_{\alpha\beta} = (R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}) = \overline{G}_{\alpha\beta} + w_{\alpha||\beta} + \frac{1}{2} w_{\alpha} w_{\beta} - \frac{1}{2} g_{\alpha\beta}\left[w^{\mu}{}_{||\mu} - \frac{1}{2} w^{2}\right], \quad (6)$$

where barred quantities refer to RST terms, and "$||$" denotes RST differentiation. Now, if one performs a conformal transformation $g_{\mu\nu} \to g'_{\mu\nu} = \Omega^{2}(x) g_{\mu\nu}$, then the connections $\Gamma^{\alpha}_{\mu\nu}$ remain invariant provided a gauge transformation $w_{\mu} \to w'_{\mu} = w_{\mu} + (\ell n\ \Omega)_{,\mu}$ is performed upon the gauge vector. Hence, if w_{μ} is the gradient of some scalar function $\Phi(x)$, a RST (with metric $g'_{\mu\nu}$) can be achieved departing from a WST (with metric $g_{\mu\nu}$) by means of a conformal transformation. A Weyl integrable space time (WIST) is characterized precisely by this feature:

$$w_{\mu}(\text{WIST}) = \Phi_{,\mu}\ . \quad (7)$$

RSTs and WISTs are thus conformally related, and via Stoke's theorem one obtains $\oint d\ell(\text{WIST}) = \ell \oint \Phi^{,\mu}\, dx_{\mu} = 0$, that is, the length of any vector is preserved around a small closed circuit, just as in the RST case (remark, however, that the connections are not the same!).

3. PALATINI'S METHOD

Starting from an arbitrary affine geometry, we shall vary independently (Palatini's variation[3]) the metric tensor $g_{\mu\nu}$ and the affine object $\Gamma^{\alpha}_{\mu\nu}$ in Einstein's Lagrangian $L_{E} = \sqrt{-g}\, R$ to obtain

$$R_{\mu\nu} = 0,\ g_{\mu\nu;\alpha} = g_{\mu\nu||\alpha} = 0\ , \quad (8)$$

that is, Einstein's equations for the vacuum and the characterization of a RST (here, not as an a priori imposition but as a consequence of a prescribed dynamical process). In summary, if we have besides L_{E} a matter term L_{M} (for any fields coupled minimally to gravity), we can state that

Minimal Coupling + Palatini's variation → Einstein's eqs. in RST. (9)

However, when we apply Palatini's method to a non-minimally coupled Lagrangian, e.g., $L_{n-m} = \sqrt{-g}\, \psi^2 R$, $\psi(x)$ being some scalar function, we end up with a WIST structure, as we shall sketch here[4]. The equations from variation are

$$\psi^2 G_{\mu\nu}=0, \quad \int \sqrt{-g}\, \psi^2 g^{\mu\nu} \gamma R_{\mu\nu} = \int \sqrt{-g}\psi^2 g^{\mu\nu} \left[\delta\Gamma^{\alpha}_{\mu\alpha;\nu} - \delta\Gamma^{\alpha}_{\mu\nu;\alpha} \right] = 0 \qquad (10)$$

where we used a coordinate system for which, locally, $\Gamma^{\alpha}_{\mu\nu} = 0$. Performing partial integrations and using[5] $\sqrt{-g}_{;\alpha} = \sqrt{-g}_{,\alpha} - \Gamma^{\lambda}_{\lambda\alpha} \sqrt{-g}$, we get that $(\sqrt{-g}\, \psi^2 g^{\mu\nu})_{;\alpha} = 0$. Since for a Christoffel symbol holds $\{^{\lambda}_{\lambda\alpha}\} = (\ell_n \sqrt{-g})_{,\alpha}$, we therefore obtain $g_{\mu\nu;\alpha} = (-\ell_n\, \psi^2)_{,\alpha}\, g_{\mu\nu} = g_{\mu\nu} W_\alpha$, where W_α is of precisely of WIST type. Hence we conclude that, indeed,

Non-minimal coupling + Palatini's variation $\rightarrow$ *Einstein's eqs. in WIST* (11)

(Remark that if $L_{n-m} = \sqrt{-g}\, (\psi^2 R + \varepsilon R_{\mu\nu} \psi^{\mu}\psi^{\nu})$, $\psi_\mu(x)$ being a vector field, one would also get[6] $(g_{\mu\nu} + \varepsilon \frac{\psi_\mu \psi_\nu}{\psi^2})_{;\alpha} = -(\ell n\, \psi^2)_{,\alpha} \times (g_{\mu\nu} + \varepsilon \frac{\psi_\mu \psi_\nu}{\psi^2})$!)

4. ELECTROMAGNETISM NON-MINIMALLY COUPLED TO GRAVITY

Let us examine now the non-minimal Lagrangian[7]

$$L_{em} = \sqrt{-g} \left[\frac{1}{k} R - R A_\mu A^\mu - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} \right] (+L_{min}\ (matter)), \qquad (12)$$

where $A_\mu(x)$ is an electromagnetic four-potential and $f_{\mu\nu} = (A_{\mu,\nu} - A_{\nu,\mu})$ is the corresponding field strength tensor Varying in Palatini's fashion, the resulting equations are

$$(\tfrac{1}{k} - A^2) G_{\mu\nu} = - E_{\mu\nu} - R A_\mu A_\nu \ (-T_{\mu\nu}\ (matter)), \qquad (13)$$

$$f^{\mu\nu}{}_{\|\nu} = - R A^\mu , \quad g_{\mu\nu;\alpha} = (-\ell n\, A^2)_{,\alpha}\, g_{\mu\nu} , \qquad (14)$$

with $E_{\mu\nu} = (f_{\mu\alpha} f^{\alpha}{}_{\nu} + \frac{1}{4} g_{\mu\nu} f_{\alpha\beta} f^{\alpha\beta})$. The source term $(-RA^\mu)$ corresponds to a (cosmological!) rest mass for the photon, and besides Einstein's equations one gets WIST charactherization, as before. Adopting the following ansatz,

$$R = 0,\ f_{\mu\nu} = 0\ (so\ E_{\mu\nu} = 0),\ A_\mu = b(t)\, \delta^0_\mu , \qquad (15)$$

there results $R_{\mu\nu} = 0$. If, also, the line element is of Friedmann--Robertson-Walker (FRW) type, such as $ds^2 = dt^2 - a^2(t)\, d\,\Omega^2$, then in the case of Euclidean section one gets the system

$$\ddot{a} = \ddot{B} + \frac{1}{2}\dot{B}^2 + 2\dot{B}\dot{a} \quad , \quad \dot{a}^2 = \dot{B}^2 + \dot{B}\dot{a} \tag{16}$$

(where $B(t) = \ln b^2$) with solutions $b(t) \sim t^{-\gamma}$, $a(t) \sim t^{\varepsilon}$, γ and ε being positive constants. These solutions describe a WIST that becomes a RST as it expands, a feature that suggests the possibility (to be examined elsewhere[6]), that different WIST and RST domains of the cosmos could be assimptotically related.

5. THE COSMOLOGICAL CONSTANT: A GENERATIVE APROACH

According to (10), if we assume that $G_{\mu\nu}(WST) = 0$ (Einstein's eqs. for an empty WST), then formally we can write

$$\bar{G}_{\mu\nu}(RST) = -(\omega_{\mu||\nu} + \frac{1}{2}\omega_\mu\omega_\nu - \frac{1}{2}g_{\mu\nu}\left[\omega^\mu{}_{||\mu} - \frac{1}{2}\omega^2\right]) = -k\,\tilde{T}_{\mu\nu} \,. \tag{17}$$

Let us try to generate, in this way, a "cosmological constant" Λ, that is, we set $k\,\tilde{T}_{\mu\nu} = -\Lambda\, g_{\mu\nu}$. Then, contracting, one obtains

$$\Lambda = \frac{3}{4}(\omega^\mu{}_{||\mu} - \frac{1}{2}\omega^2) = const. \quad , \tag{18}$$

and thus a link is established between the functions in the metric and the components of ω_μ. If, for example, we adopt a FRW line element and the ansatz $\omega_\mu = \psi(t)\,\delta^0_\mu$, then if $\psi(t) \sim exp\,\alpha t$ there results a De Sitter-type of solution, $\ln\sqrt{-g} = \ln a^3(t) \sim \gamma t$; and conversselly, if we start with an a priori exponential radius $a(t) \sim exp\,\beta t$, we end up with $\psi(t) \sim exp\,\varepsilon t$, with constants $\alpha, \varepsilon < 0$ and $\gamma, \beta > 0$. Again, WST regimes and the expansion factor of the cosmos can be assimptotically related, while a "cosmological constant" appears for the RST regime.

We conclude observing that a deep relation among non-minimal coupling and WST structures begins to be displayed, and that WISTs seem to furnish as alternative but viable model for the large-scale cosmos. Further explorations on this subject are in progress[8].

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REFERENCES

[1] Weyl, H., "Space, Time and Matter" (Dover, N.York, 1952), chap. 4, Adler, A., Bazin, M., and Schiffer, M., "Introduction to General Relativity" (McGraw-Hill, N.York, 1965), chap. 13.

[2] Rosen, N., Found.Phys. 12 nº 3 (1982), pg 213; Canuto, V., in Ruffini, R. (ed.), "Proceedings of the 2nd Marcel Grossman Meeting" (North Holland, 1982), pg 729.

[3] Anderson, J.L., "Principles of General Relativity" (Academic Press, N.York, 1980), pg. 361.

[4] Novello, M., and Heintzmann, H., Phys.Letts. 109A nº 9 (1985),pg 4.

[5] Misner, C.W., Thorne K.S., and Wheeler, J.A., "Gravitation" (Freeman, S.Francisco, 1973), chap. 8.

[6] Novello, M., and Oliveira, L.A.R., in preparation.

[7] Novello, M., and Salim, J.M., Fund.Cosm.Phys. 8 nº 3 (1983), pg 201.

[8] Oliveira, L.A.R., PhD Thesis, in preparation.