ISSN 0029 - 3865

CBPF-NF-062/85

A NON-ABELIAN AHARONOV-BOHM EFFECT IN THE FRAMEWORK OF FEYNMAN PSEUDO-CLASSICAL PATH INTEGRALS

by

LUIZ C.L. BOTELHO and J.C. DE MELLO*

Centro Brasileiro de Pesquisas Físicas - CNPq/CBPF
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

*Instituto de Física
Universidade Federal do Rio de Janeiro
Cidade Universitária - Ilha do Fundão
21945 - Rio de Janeiro, RJ - Brasil

ABSTRACT

We analyze the non-abelian Aharonov-Bohm effect in the Feynman Path Integrals framework generalized to pseudo-classical mechanics.

Key-words: Aharonov-Bohm effect; Pseudo-classical mechanics; Feynman Path Integrals; Non-abeliam gauge theory.

The non-abelian generalization of the Aharonov-Bohm effect [1] was proposed by Wu and Yang [2] in 1975. In a recent comment [3] we analized this non-abelian effect by exploiting the existence of internal degrees of freedom associated to the particles colour charge.

Our aim in this comment is to generalize the classical analysis implemented in Ref. [3] to the quantum level by using the Feynman path integrals framework generalized to pseudo-classical mechanics.

Let us start our analysis by recalling some basic facts of pseudo-classical mechanics. In this framework the motion of a colour (spinless) massive charged particle is characterized by the usual (relativistic) vector position $x^{\mu}(\xi)$ added to a set of Grassman complex variable $\{\Theta_{\ell}(\xi),\ \Theta^{*}_{\ell}(\xi)\}$, where $\xi_i \leq \xi \leq \xi_f$ denotes a parameter describing the evolution of the system and $\ell = 1,---,N$ is the gauge group number of generators $\{\lambda_i\}$ [5], The pseudo-classical lagrangean of a spinless colour particle in the presence of an external Yang-Mills gauge field $A_{\mu}(X) = A^{i}_{\mu}(X)\lambda_i$ is given by [4]:

$$\mathcal{L}(X^{\mu}(\xi),\Theta_{\ell}(\xi),\Theta^{*}_{\ell}(\xi),A^{i}_{\mu}(X)) = -mc\sqrt{\dot{X}^2(\xi)} + \frac{1}{2}\, i \sum_{\ell=1}^{N} (\Theta^{*}_{\ell}\dot{\Theta}_{\ell} - \dot{\Theta}^{*}_{\ell}\Theta_{\ell})(\xi) -$$

$$-g(\Theta^{*}_{\ell}(\lambda_i)_{\ell k}\Theta_k)(\xi)\ A^{i}_{\mu}(X(\xi))\ \dot{X}^{\mu}(\xi) \qquad (1)$$

In the Feynman path integral formalism for quantization, the quantum propagation of the above physical system is given by the "continuous" sum of all trajectories connecting the initial and final states characterized respectively by the initial vector position $X_\mu(\xi_i) = X_\mu$ and the colour degree $\Theta_j(\xi_i)=\beta_j$ and $X_\mu(\xi_f)=Y_\mu$ with $\Theta^*_k(\xi_f)=\eta^*_k$. We propose for this continuous sum the following path integral:

$$G(|X_\mu,\beta_f>;|y_\mu,\eta^*_k>) = \int_{\substack{X_\mu(\xi_i)=X_\mu \\ X_\mu(\xi_f)=Y_\mu}} \Big(\prod_{\xi_i<\xi<\xi_f} d\,X_\mu(\xi)\Big)\Big(\prod_{\ell=1}^{N} \prod_{\xi_i<\xi<\xi_f} d\Theta_\ell(\xi)\; d\Theta^*_\ell(\xi)\Big)\,.$$

$$.\,\Big(\Theta_j(\xi_i)\Theta^*_k(\xi_f)\Big).\ \exp\left\{(i/\hbar)\int_{\xi_i}^{\xi_f} \mathcal{L}\Big(X^\mu(\xi),\ \Theta_\ell(\xi),\ \Theta^*_\ell(\xi),\ A^i_\mu(X)\Big)\ d\xi\right\}, \qquad (2)$$

where $\mathcal{L}(X^\mu(\xi),\ \Theta_\ell(\xi),\ \Theta^*_\ell(\xi),\ A^i_\mu(X))$ is given by eq.(1).

We note the similarity of the grassmanian degrees propagation with the usual quantum field fermion propagation and the eq.(2) posseses group matrix indices. The remarkable feature of the proposed grassmanian path measure is that we can evalu ate it explicitly and leading to the well-known Wu-Yang factor as we show below, result which generalizes the Feynman's result [6] on the abelian case. This feature stems from the fact that the grassmanian path integrals are of Gaussian type. So, we have to evaluate the expression:

$$I_{jk}\Big(X_\mu(\xi),A^i_\mu(X)\Big) = \int \Big(\prod_{\ell=1}^{N} \prod_{\xi_i<\xi<\xi_f} d\Theta_\ell(\xi)\ d\,\Theta^*_\ell(\xi)\Big).\Big(\Theta_j(\xi_i).\Theta^*_k(\xi_f)\Big)\ .$$

$$.\exp\left\{(i/\hbar)\int_{\xi_i}^{\xi_f} \frac{1}{2}\, i \sum_{\ell=1}^{N} (\Theta^*_\ell\,\dot{\Theta}_\ell - \dot{\Theta}^*_\ell\,\Theta_\ell)(\xi) - g\Big(\Theta^*_\ell(\lambda_i)_{\ell k}\Theta_k\Big)(\xi)A^i_\mu\Big(X(\xi)\Big)\dot{X}^\mu(\xi)\right\} \qquad (3)$$

By noting the useful identity $(\Theta_\ell^* \dot{\Theta}_\ell - \dot{\Theta}_\ell^* \Theta_\ell)(\xi) = -\frac{d}{d\xi}(\Theta_\ell \Theta_\ell^*)(\xi) - 2(\dot{\Theta}_\ell^* \Theta_\ell)(\xi)$, and the result that the "unidimensional" Grassmanian Green's function is given by the "non-relativistic" propagator $\left(\frac{d}{d\tau}\right)^{-1}(\Sigma_1, \Sigma_2) = \theta(\Sigma_1 - \Sigma_2)$, we can evaluate exactly the expression (3) ([8] . See eq(2.20)):

$$I_{jk}\left(X_\mu(\xi);\ A^i_\mu\ (X)\right) = e^{(1/2k)\ \left[(\eta_k \eta_k^*) - (\beta_j \beta_j^*)\right]} \ . \\ . \left(\mathbb{P}\left\{ e^{ig \int_{\xi_i}^{\xi_f} A^i_\mu\ \left(X_\mu(\xi)\right)\ .\ \dot{X}^\mu(\xi)\,\lambda_i} \right\}\right)_{jk} \tag{4}$$

So, we obtain the expression for the proposed propagator:

$$G\left(|X_\mu,\ \beta_j>;\ |Y_\mu,\ \eta_k^*>\right) = \int \prod_{\substack{\xi_i<\xi<\xi_f \\ X_\mu(\xi_i)=X_\mu \\ X_\mu(\xi_f)=Y_\mu}} dX_\mu(\xi)\ e^{i/\hbar \left\{ \int_{\xi_i}^{\xi_f} - mc\sqrt{\dot{X}^2(\xi)}\, d\xi \right\}}$$

$$.\ I_{jk}\left(X_\mu(\xi),\ A^i_\mu(X)\right). \tag{5}$$

Now let us analyze the following "Gedanken" experiment: ([3]) a quantum charged U(N), spinless particle propagates in a region R where there is a non-zero Yang-Mills gauge field with the property that its associated field strength vanishes, with the initial and final states $|X_\mu, \beta_j>; |Y_\mu, \eta_k^*>$. We remark that this region can have an arbitrary topology. The associated quantum probability amplitude is given by Eq. (5) i.e,, $G(|X_\mu,\ \beta_j>;\ |Y_\mu,\ \eta_k^*>)$.

Now we observe the following fundamental fact: the only system trajectory that contributes to the Wu-Yang factor in (5) comes from the classical trajectory, due to the vanishes field strength condition, as we can see by considering the functional Taylor expansion of (4) around the classical trajectory $\{X^{\mu}_{cl}(\xi);\ \xi_i \leq \xi \leq \xi_f\}$:

$$\left[I\left(X^{\mu}_{cl}(\xi) + X^{\mu}_{q}(\xi);\ A^{i}_{\mu}(X)\right)\right] - \left[I\left(X^{\mu}_{cl}(\xi);\ A^{i}_{\mu}(X)\right)\right] =$$

$$= \sum_{N=1} \frac{1}{N!} \int_{\xi_i}^{\xi_f} d\eta_1 \left(X_q^{\alpha_1}(\eta_1)\right) --- \int_{\xi_i}^{\xi_f} d\eta_N \left(X_q^{\alpha_N}(\eta_N)\right) \left. \frac{\delta^{(N)} I\left[Z_{\mu}(\xi)\right]}{\delta X_q^{\alpha_1}(\eta_1) --- \delta X_q^{\alpha_N}(\eta_N)} \right|_{Z_{\mu}(\xi) = X^{\mu}_{cl}(\xi)} \tag{6}$$

Since all these functional derivatives in Eq. (6) are multiplicatively proportional to the field strength (see for instance Eq. (20), Eq. (21) of Ref. [8]); we achieve the above quoted result.

Now we consider the reversal quantum propagation, characterized by the initial and final states $|Y_{\mu}, \eta_k>$ and $|X_{\mu};\ \beta^*_j> :\ G\left(|Y_{\mu}, \eta_{\mu}>;\ |X_{\mu};\ \beta^*_j>\right)$. As usual, we would expect that the probability amplitude for the "closed circuit" $\left(|X_{\mu}, \eta_k>;\ |X_{\mu}, \beta^*_j>\right)$ given by:

$$\sum_{\{K\}} G\left(|X_{\mu}, \beta_j>;\ |Y_{\mu}, \eta^*_k>\right) G\left(|Y_{\mu}, \eta_k>; |X_{\mu}, \beta^*_j>\right)$$

should be a phase independent of the particular system trajectory connecting the above initial and final states. But due to the colour degrees (see Eq. (5) - Eq. (6)) we obtain that this phase is given by the Wu-Yang factor defined solely by the classical closed system trajectory.

So, we have shown that the outcome of the propose "Gedanken" experiment, analysed on the quantum level, depends fundamentally on the (matrix elements) of the Wu-Yang non-integrable factor as obtained in our previous study (Ref. [3]) and predicted earlier by Wu and Yang.

Note added: after this comment was completed, we received the pre-print CPT-85/P-1769 by P.A. Horvathy, where physical applications of the above mentioned non-abelian Aharonov-Bonn effect is made.

ACKNOWLEDGMENT

We would like to thank Professor P.A. Horvathy for seding us, kindly, the prior publication, of pre-print CPT-85/P.1769 (to be published in Phys. Rev. D) and to the financial support by CNPq Brasil.

REFERENCES

[1] Y. Aharonov and D. Bohm - Phys Rev 115 (1959) 485.

[2] P.A. Horváthy and J. Kollar - (1984) - Preprint-Marseille CPT84/P 1663.

[3] Luiz C.L. Botelho and J.C. de Mello - J. Phy A 18 (1985).

[4] A. Barducci; R. Casalbuoni and L. Lusanna-Nucl.Phys B 124. (1977) 93.

[5] F.A. Berezin and M.S. Marinov - Ann. Phys. 104 (1977) 336.

[6] R.P. Feynman and A.R. Hibbs-Quantum Mechanics and Path integrals (Mc Graw-Hill-New York-1965).

[7] J.L. Gervais and A. Neveu - Nucl. Phys B 163 (1980) 189.

[8] J.L. Gervais and A. Neveu - Nucl. Phys B 153 (1979) 445.

[9] T.T. Wu, C.N. Yang, Phys. Rev. D12, 3845 (1975).