From modular invariants to graphs: the modular splitting method

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Abstract

We start with a given modular invariant $\mathcal{M}$ of a two dimensional $\widehat{su}(n)_k$ conformal field theory (CFT) and present a general method for solving the Ocneanu modular splitting equation and then determine, in a step-by-step explicit construction, 1) the generalized partition functions corresponding to the introduction of boundary conditions and defect lines; 2) the quantum symmetries of the higher ADE graph G associated to the initial modular invariant $\mathcal{M}$. Notice that one does not suppose here that the graph G is already known, since it appears as a by-product of the calculations. We present a standard example belonging to the $\widehat{su}(2)_k$ family and analyze several $\widehat{su}(3)_k$ exceptional cases at levels 5 and 9.

Keywords: conformal field theory, modular invariance, higher ADE systems, fusion algebra, Hopf algebra, quantum groupoïds.

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1 Introduction

Following the works of [18], it was shown that to every modular invariant of a 2d CFT one can associate a special kind of quantum groupoïd $\mathcal{B}(G)$, constructed from the combinatorial and modular data [13] of a graph G [23, 7, 25, 27, 10]. This quantum groupoïd $\mathcal{B}(G)$ plays a central role in the classification of 2d CFT, since it also encodes information on the theory when considered in various environments (not only on the bulk but also with boundary conditions and defect lines): the corresponding generalized partition functions are expressed in terms of a set of non-negative integer coefficients that can be determined from associative properties of structural maps of $\mathcal{B}(G)$ [28, 24, 25]. A series of papers [4, 23, 5, 6, 25] present the computations allowing to obtain these coefficients from a general study of the graph G and its quantum symmetries. In this approach, the set of graphs G is taken as an input. For the $\widehat{su}(2)_k$ model, the graphs G are the ADE Dynkin diagram, and for the $\widehat{su}(3)_k$ the Di Francesco-Zuber diagrams. A list of graphs has also been proposed in [20] for the $\widehat{su}(4)_k$ model. For a general $SU(N)$ system, the set of graphs G presents the following pattern. There is always the infinite series of $\mathcal{A}_k$ graphs, which are the truncated Weyl alcoves at some level k of $SU(N)$ irreps. Other infinite series are obtained by orbifolding and conjugation methods, but there are also some exceptional graphs (generalizing the E_6 and E_8 diagrams of the $SU(2)$ series), that can not be obtained in that way (to some extent, the E_7 diagram can be obtained from a careful study of the D_{10} case). One of the purposes of this article is actually to present a method to obtain these graphs.

We start with a modular invariant of a 2d $\widehat{su}(n)_k$ CFT as initial data. Classification of modular invariants is only completed for $n = 2$ and 3, but there exist several algorithms, mostly due to T. Gannon, that allow one to obtain modular invariants up to rather high levels of any affine algebra. By solving the modular splitting equation (to be recalled later), we obtain the coefficients of the generalized partition functions, as well as the quantum symmetries of the graph G, encoded in the Ocneanu graph $Oc(G)$. The graph G itself is then obtained at this stage as a subgraph or a module graph of its own quantum symmetry graph: it appears as a by-product of the computations.

Notice that the determination of the higher ADE graphs G by solving the modular splitting equation seems to be the method followed by A. Ocneanu (see [19]) to obtain the lists of $SU(3)$ and $SU(4)$ graphs presented in [20], but explicitation of his method was never been made available in the literature. The method that we describe here (that incorporates the solution of the modular splitting equation itself) was briefly presented in [8] for the study of the non simply laced diagram F_4, and is presented here in more general grounds.

The paper is organized as follows. In section **2** we review some results of CFT in order to fix our notations, and present the basic steps of the method allowing to solve the modular splitting equation. Section **3** treats with more technical details of the resolution, making the difference between commutativity or non commutativity of the quantum symmetry algebra. In the last section we analyze some examples in order to illustrate the techniques. First we treat the E_6 modular invariant of the $SU(2)$ family; then two exceptional $SU(3)$ modular invariants at level 5, labelled by the graphs $\mathcal{E}_5$ and $\mathcal{E}_5/3$. The last example is the level 9 exceptional $SU(3)$ modular invariant, which is a special case for two reasons. The first one is that it leads to a non-commutative algebra of quantum symmetries. The second reason is that in the first list of $SU(3)$ graphs proposed in [11], three different graphs were associated to this modular invariant. With the methods presented in this paper, we only find two graphs associated to it, in accordance with the final list of Ocneanu published in [20].

2 CFT and graphs

Consider a 2d CFT defined on a torus, where the chiral algebra is an affine algebra $\widehat{su}(n)_k$ at level k. The modular invariant partition function reads

$$\mathcal{Z} = \sum_{\lambda,\mu} \chi_\lambda \, \mathcal{M}_{\lambda\mu} \, \overline{\chi_\mu} \, , \tag{1}$$

where χ_λ is the character of the element λ of the finite set of integrable representations of $\widehat{su}(n)_k$, and where the matrix $\mathcal{M}$ is called the modular invariant: it commutes with the generators $\mathcal{S}$ and $\mathcal{T}$ of the modular group $PSL(2,\mathbb{Z})$. The introduction of boundary conditions (labelled by a,b), defect lines (labelled by x,y) or the combination of both, result in the following generalized partition functions (see [3, 1, 24]):

$$\mathcal{Z}_{a|b} = \sum_{\lambda} (\mathcal{F}_\lambda)_{ab} \chi_\lambda \tag{2}$$

$$\mathcal{Z}_{x|y} = \sum_{\lambda,\mu} (\mathcal{V}_{\lambda\mu})_{xy} \chi_\lambda \overline{\chi_\mu} \tag{3}$$

$$\mathcal{Z}_{x|ab} = \sum_{\lambda} (\mathcal{F}_\lambda \, S_x)_{ab} \, \chi_\lambda \tag{4}$$

All coefficients appearing in the above expressions express multiplicities of irreducible representations in the Hilbert space of the corresponding theory and are therefore non-negative integers. They are conveniently encoded in a set of matrices: the annular matrices F_λ with coefficients $(\mathcal{F}_\lambda)_{ab}$; the double annular matrices $V_{\lambda\mu}$ with coefficients $(\mathcal{V}_{\lambda\mu})_{xy}$ and the dual annular matrices S_x with coefficients $(S_x)_{ab}$. The different set of indices run as $\lambda,\mu = 0,\ldots,d_I - 1$; $a,b = 0,\ldots,d_G - 1$ and $x,y = 0,\ldots,d_O - 1$. The integer d_I is the number of irreps at the given level k; d_G and d_O are given in terms of the modular invariant $\mathcal{M}$ by $d_G = Tr(\mathcal{M})$ and $d_O = Tr(\mathcal{M}\mathcal{M}^\dagger)$ (see [21, 2, 12]).

Compatibilities conditions – in the same spirit than those defined by Cardy in [3] for boundary conditions – impose relations on the above coefficients (see [22, 24]). Altogether they read:

$$F_\lambda \, F_{\lambda'} = \sum_{\lambda''} \mathcal{N}_{\lambda\lambda'}^{\lambda''} \, F_{\lambda''} \tag{5}$$

$$V_{\lambda\mu} \, V_{\lambda'\mu'} = \sum_{\lambda''\mu''} \mathcal{N}_{\lambda\lambda'}^{\lambda''} \, \mathcal{N}_{\mu\mu'}^{\mu''} \, V_{\lambda''\mu''} \tag{6}$$

$$S_x \, S_y = \sum_{z} \mathcal{O}_{yx}^{z} \, S_z \tag{7}$$

$\mathcal{N}_{\lambda\mu}^{\nu}$ are the fusion coefficients describing the tensor product decomposition $\lambda \star \mu = \sum_\nu \mathcal{N}_{\lambda\mu}^{\nu} \, \nu$ of representations λ and μ of $\widehat{su}(n)_k$. They can be encoded in matrices N_λ called fusion matrices. $\mathcal{O}_{xy}^{z}$ are the quantum symmetry coefficients and can be encoded in matrices O_x called quantum symmetry matrices.

The matrices $\{F_\lambda, N_\lambda, O_x, V_{\lambda\mu}, S_x\}$ have non negative integer coefficients: they can be seen as the adjacency matrices of a set of graphs. Knowledge of these graphs helps therefore to the complete determination of the different partition functions. All these coefficients also define (or can be obtained by) structural maps of a special kind of quantum groupoïd [18, 23, 25, 7, 10]. It is not the purpose of this paper to explore those correspondences, nor to study the mathematical aspects of this quantum groupoïd. What we will do here is to determine, taking as initial data the knowledge of the modular invariant $\mathcal{M}$, all the coefficients of the above matrices.

2.1 Steps of the resolution

We start with the double fusion equations (6), which are matrix equations involving the double annular matrices $V_{\lambda\mu}$, of size $d_O \times d_O$, with coefficients $(V_{\lambda\mu})_{xy}$. Notice that these coefficients can also be encoded in matrices W_{xy}, of size $d_I \times d_I$, with coefficients $(W_{xy})_{\lambda\mu} = (V_{\lambda\mu})_{xy}$. The W_{xy} are called double toric matrices. When no defect lines are present ($x = y = 0$), we must recover the modular invariant of the theory, therefore $W_{00} = \mathcal{M}$. Using the double toric matrices W_{xy}, the set of equations (6) read:

$$\sum_z (W_{xz})_{\lambda\mu} \, W_{zy} = N_\lambda \, W_{xy} \, N_\mu^{tr} \, . \tag{8}$$

The successive steps of resolution are the following:

Step 1: toric matrices Setting $x = y = 0$ in (8) and using the fact that $W_{00} = \mathcal{M}$ we get:

$$\sum_z (W_{0z})_{\lambda\mu} \, W_{z0} = N_\lambda \, \mathcal{M} \, N_\mu^{tr} \, . \tag{9}$$

This equation was first presented by A. Ocneanu in [20] and is called the **modular splitting equation**. The r.h.s. of (9) involves only known quantities, namely the modular invariant $\mathcal{M}$ and the fusion matrices N_λ. The l.h.s. involves the set of toric matrices W_{z0} and W_{0z}, that we determine from this equation.

Step 2: double fusion matrices Setting $y = 0$ in (8) we get:

$$\sum_z (W_{xz})_{\lambda\mu} \, W_{z0} = N_\lambda \, W_{x0} \, N_\mu^{tr} \tag{10}$$

Once the toric matrices W_{x0} have been determined from Step 1, the r.h.s. of (10) then involves only known quantities. Resolution of these equations determine the double toric matrices W_{xy} – and equivalently the double fusion matrices $V_{\lambda\mu}$ – appearing in the l.h.s. of (10).

Step 3: Ocneanu graph The double fusion matrices $V_{\lambda\mu}$ are generated by a subset of fundamental matrices V_{f0} and V_{0f}, where f stands for the generators of the fusion algebra (for $SU(n)$ there are $n-1$ fundamental generators). These matrices are the adjacency matrices of a graph called the Ocneanu graph. Its graph algebra is the quantum symmetry algebra, encoded in the set of matrices O_x.

Step 4: higher ADE graph G The higher ADE graph G corresponding to the initial modular invariant $\mathcal{M}$ is recovered at this stage as a module graph of the Ocneanu graph. It may be a subgraph of $Oc(G)$ or an orbifold of one of its subgraphs. One also distinguishes type I cases (also called subgroup or self-fusion cases) and type II cases (also called module or non self-fusion cases).

3 From the modular invariant to graphs

We start with a modular invariant $\mathcal{M}$ at a given level k of a $\widehat{su}(n)$ CFT, and the corresponding fusion matrices N_λ.

3.1 Determination of toric matrices W_{x0}

We compute the set of matrices $K_{\lambda\mu}$ defined by:

$$K_{\lambda\mu} = N_\lambda \, \mathcal{M} \, N_\mu^{tr} \ . \tag{11}$$

The modular splitting equation (9) then reads:

$$K_{\lambda\mu} = \sum_{z=0}^{d_O - 1} (W_{0z})_{\lambda\mu} \, W_{z0} \ . \tag{12}$$

This equation can be viewed as the linear expansion of the matrix $K_{\lambda\mu}$ over the set of toric matrices W_{z0}, where the coefficients of this expansion are the non-negative integers $(W_{0z})_{\lambda\mu}$. The number d_O is the dimension of the Ocneanu quantum symmetry algebra, it is evaluated by $d_O = Tr(\mathcal{M}\mathcal{M}^\dagger)$. The algebra of quantum symmetries comes with a basis (call its elements z) which is special because structure constants of the algebra, in this basis, are non-negative integers. We introduce the linear map from the space of quantum symmetries to the space of $d_I \times d_I$ matrices defined by $z \mapsto W_{z0}$. This map is not necessarily injective: although elements z of the quantum symmetries are linearly independent, it may not be so for the toric matrices W_{z0} (in particular two distinct elements of the quantum symmetries can sometimes be associated with the same toric matrix). Let us call r the number of linearly independent matrices W_{z0}. Equation (12) tells us that each $K_{\lambda\mu}$ (a matrix), defined by (11), can be decomposed on the r dimensional vector space spanned by the vectors (matrices) W_{z0}. r can be obtained as follows. From (11) we build a matrix K with elements of the form $\mathrm{K}_{\{\lambda\mu\},\{\lambda'\mu'\}}$, which means that each line of K is a flattened[1] matrix $K_{\lambda\mu}$. Then r is obtained as the (line) rank of the matrix K, since the rank gives precisely the maximal number of independent lines of K, therefore the number r of linearly independent matrices W_{z0}. Two cases are therefore to be considered: depending if toric matrices are all linearly independent (the map $z \mapsto W_{z0}$ is injective and $r = d_O$) or not ($r < d_O$).

We also introduce a scalar product in the vector space of quantum symmetries for which the z basis is orthonormal. We consider the squared norm of the element $\sum_z (W_{0z})_{\lambda\mu} z$ and denote it $||K_{\lambda\mu}||^2$. This is an abuse of notation, "justified" by equation (12), and in the same

[1]By flattened matrix we mean that if $K_{\lambda\mu} = \begin{pmatrix} a & .. & b \\ .. & .. & .. \\ c & .. & d \end{pmatrix}$, then the flattened matrix is $(a \,..\, b \,..\,..\,..\, c \,..\, d)$.

way, we shall often talk, in what follows, of the "squared norm of the matrix $K_{\lambda\mu}$", therefore identifying z with W_{z0}, although the linear map is not necessarily an isomorphism. We have the following property:

Property 1 *The squared norm of the matrix* $K_{\lambda\mu}$ *is given by:*

$$||K_{\lambda\mu}||^2 = (K_{\lambda\mu})_{\lambda^*\mu^*} \ . \tag{13}$$

Proof: We have:

$$\begin{aligned} ||K_{\lambda\mu}||^2 &= \sum_z |(W_{0z})_{\lambda,\mu}|^2 \\ &= \sum_z (W_{0z})_{\lambda\mu} \, (W_{z0})_{\lambda^*\mu^*} \\ &= (K_{\lambda\mu})_{\lambda^*\mu^*} \end{aligned}$$

From the first to the second line we used the following property:

$$(W_{0z})_{\lambda\mu} = (W_{z0})_{\lambda^*\mu^*} \tag{14}$$

that can be derived from the relation $V_{\lambda^*\mu^*} = (V_{\lambda\mu})^{tr}$, where λ^* is the conjugated irrep of λ (see [23]). From the second to the third line we use Eq. (12) in matrix components. ∎

We now treat the two cases to be considered. Note: an explicit study of all cases seems to indicate that the linear independence (or not) of the toric matrices reflects the commutativity (or not) of the quantum symmetry algebra.

Non-degenerate case $r = d_O$. This happens when all toric matrices W_{z0} are linearly independent. The set of $K_{\lambda\mu}$ matrices are calculated from the initial data $\mathcal{M}$ and N_λ from (11). The determination of the toric matrices W_{z0} are recursively obtained from a discussion of the squared norm of matrices $K_{\lambda\mu}$, directly obtained from (13), which has to be a sum of squared integers.

- Consider the set of linearly independent matrices $K_{\lambda\mu}$ of squared norm 1. From (12) the solution is that each such matrix is equal to a toric matrix W_{z0}.
- Next we consider the set of linearly independent matrices $K_{\lambda\mu}$ of squared norm 2. In this case from (12) each such matrix is equal to the sum of two toric matrices. We have three cases: (i) $K_{\lambda\mu}$ is equal to the sum of two already determined toric matrices (no new information); (ii) it is the sum of an already determined toric matrix and of a new one; (iii) it is equal to the sum of two new toric matrices. To distinguish from cases (ii) and (iii), we calculate the set of differences $K_{\lambda\mu} - W_i$ where W_i runs into the set of determined toric matrices, and check if the obtained matrix has non-negative integer coefficients: in this case we determine a new toric matrix given by $K_{\lambda\mu} - W_i$.
- Next we consider the set of linearly independent matrices $K_{\lambda\mu}$ of squared norm 3. From (12) each such matrix is equal to the sum of three toric matrices. Either (i) $K_{\lambda\mu}$ is equal to the sum of three already determined toric matrices; (ii) it is equal to the sum of a determined toric matrix and of two new ones; (iii) it is equal to the sum of two

already determined matrices and a new one; or (iv) it is equal to the sum of three new toric matrices. We calculate the set of differences $K_{\lambda\mu} - W_i$ and $K_{\lambda\mu} - W_i - W_j$ where W_i, W_j runs into the set of determined toric matrices, and check whenever the obtained matrix has non-negative integer coefficients.

- For the set of linearly independent matrices $K_{\lambda\mu}$ of squared norm 4 there are two possibilities. Either $K_{\lambda\mu}$ is the sum of four toric matrices, either it is equal to twice a toric matrix. In the last case, the matrix elements of $K_{\lambda\mu}$ should be either 0 or a multiple of 2, and the new toric matrix is obtained as $K_{\lambda\mu}/2$. If not, a similar discussion as the one made for the previous items allows the determination of the new toric matrices.
- The next step is to generalize the previous discussions for higher values of the squared norm, in a straightforward way.

Once the set of toric matrices W_{z0} is determined, we can of course use equation (9) to check the results.

Degenerate case $r < d_O$. The integer r may be strictly smaller than d_O: this happens when toric matrices W_{z0} are *not* linearly independent. In order to better illustrate what has to be done in this case, let us treat a "virtual" example. Suppose the dimension of the Ocneanu algebra is $d_O = 3$, and call z_1, z_2, z_3 the basis elements. The corresponding toric matrices are $W_{z_1}, W_{z_2}, W_{z_3}$, and suppose they are not linearly independent. For example let us take $W_{z_3} = W_{z_1} + W_{z_2}$, in this case we have $r = 2 < d_O$. We still use the same scalar product in the algebra of quantum symmetries, and the norm of z_3 is of course 1, but, because of the abuse of langage and notation already made before, we shall say that the "squared norm" of W_{z_3} is equal to 1 (and not 2, of course!). The problem arising from the fact that toric matrices may not be linearly independent, so that the linear expansion (12) of $K_{\lambda\mu}$ over the family of toric matrices may be not unique, can be solved by considering the squared norm of $K_{\lambda\mu}$. Continuing with our virtual example, we could hesitate between writing $K_{\lambda\mu} = W_{z_1} + 2W_{z_2}$ or $K_{\lambda\mu} = W_{z_2} + W_{z_3}$, since $W_{z_3} = W_{z_1} + W_{z_2}$. In the first case the corresponding squared norm would be 5, and in the second case it would be 2. In all cases we have met, the knowledge of the squared norm of $K_{\lambda\mu}$ from equation (13) is sufficient to bypass the ambiguity and obtain the correct linear expansion. The determination of the toric matrices can then be done step by step, in the same way as we did in the non degenerate case, starting from squared norm 1 to higher values. We refer to the $\widehat{su}(3)$ case at level 9 treated in the next section for more technical details.

3.2 Determination of double toric matrices W_{xy}

Once we have determined the toric matrices W_{x0}, we calculate the following set of matrices:

$$K^x_{\lambda\mu} = N_\lambda \, W_{x0} \, N^{tr}_\mu \tag{15}$$

Then equation (10) reads:

$$K^x_{\lambda\mu} = \sum_z (W_{xz})_{\lambda\mu} W_{z0} \; . \tag{16}$$

This equation can be viewed as the linear expansion of the matrix $K^x_{\lambda\mu}$ over the set of toric matrices W_{z0}, where the coefficients of this expansion are the non-negative integers $(W_{xz})_{\lambda\mu}$, that we want to determine. In the non degenerate case, toric matrices W_{z0} are linearly independent, the decomposition (16) is unique and the calculation is straightforward. In the degenerate case, some care has to be taken since toric matrices W_{z0} are not linearly independent: the expansion (16) is therefore not unique. Some coefficients may remain free and one needs further information to a complete determination (see next subsection).

The coefficients $(W_{xz})_{\lambda\mu}$ can also be encoded in the double fusion matrices $V_{\lambda\mu}$, that satisfy the double fusion equations (6). Setting $\mu = \mu' = 0$, $\lambda = \lambda' = 0$ and $\lambda' = \mu = 0$ respectively in Eq. (6) gives:

$$
\begin{aligned}
V_{\lambda 0}\, V_{\lambda' 0} &= \sum_{\lambda''} N_{\lambda\lambda'}^{\lambda''}\, V_{\lambda'' 0} \ , && (17)\\
V_{0\mu}\, V_{0\mu'} &= \sum_{\mu''} N_{\mu\mu'}^{\mu''}\, V_{0\mu''} \ , && (18)\\
V_{\lambda\mu'} &= V_{\lambda 0}\, V_{0\mu'} = V_{0\mu'}\, V_{\lambda 0} \ . && (19)
\end{aligned}
$$

From Eqs.(17) and (18), we see that the set of matrices $V_{\lambda 0}$ and $V_{0\lambda}$ satisfy the fusion algebra. These matrices can therefore be determined using these equations from the subset of matrices V_{f0} and V_{0f}, where f stands for the fundamental generators of the fusion algebra. For $\widehat{su}(2)$, there is one generator $f = 1$, while for $\widehat{su}(3)$, there are two conjugated generators $(1,0)$ and $(0,1)$. The determination of double fusion matrices is reduced, by the use of Eqs. (17–19), to the determination of the generators V_{f0} and V_{0f}. It is therefore sufficient to solve Eq. (16) only for the pair of indices $(\lambda\mu) = (f0)$ and $(\lambda\mu) = (0f)$, and then use Eqs. (17–19), which simplifies a lot the computational task.

3.3 Determination of the Ocneanu algebra O_x

The matrices V_{f0} and V_{0f} are the adjacency matrices of the Ocneanu graph. We denote $O_{f_L} = V_{f0}$ and $O_{f_R} = V_{0f}$, where f_L and f_R now stands for the left and right generators of the Ocneanu quantum symmetry algebra. For $SU(n)$, there are $n-1$ generators f of the fusion algebra, and therefore $2(n-1)$ generators of the quantum symmetry algebra. The Ocneanu graph is also the Cayley graph of multiplication by these generators. From the multiplication by these generators, we can reconstruct the full table of multiplication of the quantum symmetry algebra (with elements denoted x, y, z)

$$
x\, y = \sum_z \mathcal{O}_{xy}^z\, z \ . \qquad (20)
$$

This multiplication table is encoded in the "quantum symmetry matrices" O_x, which are the graph algebra matrices of the Ocneanu graph, with coefficients $(O_x)_{yz} = \mathcal{O}_{xy}^z$. They satisfy the following relations (take care with the order of indices since the quantum symmetry algebra may be non commutative):

$$
O_x\, O_y = \sum_z (O_y)_{xz}\, O_z \ . \qquad (21)
$$

Once the generators $O_{f_L} = V_{f0}$ and $O_{f_R} = V_{0f}$ have been determined from the previous step, all quantum symmetry matrices can be computed from (21).

In the degenerate case the determination of the double toric matrices W_{xy} from equation (16) is not straightforward, some coefficients being still free. A solution to this problem is provided by an analysis of the structure of the Ocneanu graph itself, since it must satisfy some conjugation and chiral conjugation properties (we refer to the level 9 $\widehat{su}(3)$ example treated in the next section for further details). Further compatibility conditions have also to be satisfied and can be used to check the results, or to determine the remaining coefficients (for degenerate cases). One of these conditions read [9]:

$$O_x\, V_{\lambda\mu} = V_{\lambda\mu}\, O_x = \sum_z (V_{\lambda\mu})_{xz}\, O_z \ . \tag{22}$$

A special case of this equation, for $x = 0$, being:

$$W_{yy'} = \sum_z (O_z)_{yy'}\, W_{0z} \ . \tag{23}$$

3.4 Determination of the higher ADE graph G

For any $\widehat{su}(n)$ at level k, we can always consider the infinite series of $\mathcal{A}_k$ graphs, which are the truncated Weyl alcoves at level k of $SU(n)$ irreps. Other infinite series are obtained by orbifolding ($\mathcal{D}_k = \mathcal{A}_k/p$) and conjugation ($\mathcal{A}_k^*, \mathcal{D}_k^*$) methods, but there are also some exceptional graphs that can not be obtained in that way. Even after using the fact that graphs have to obey a list of requirements (such as conjugation, N-ality, spectral properties and that G must be an $\mathcal{A}_k$ module) listed in [11], one needed to use some good "computer aided flair" to find them. In this "historical approach", the problem of determining the algebra of quantum symmetries $Oc(G)$ was not addressed and this algebra was even less used as a tool to determine G itself. The procedure described here is different. Starting from the modular invariant, one solves the modular splitting equations (as explained in the previous section) and determines directly the algebra of quantum symmetries $Oc(G)$, without knowing what G itself can be. Then one uses the fact that G should be *both* an $\mathcal{A}_k$ module *and* an $Oc(G)$ module (see comments in [9]). Denoting λ an element of the fusion algebra, the first module property reads $\lambda\, a = \sum_b (F_\lambda)_{ab}\, b$, with coefficients encoded by the annular matrices F_λ. The associativity property $(\lambda\, a)\, b = \lambda\, (a\, b)$ imposes the annular matrices to satisfy the fusion algebra (see Eq.(5)). Denoting x an element of the quantum symmetry algebra, the second module property reads $x\, a = \sum_b (S_x)_{ab}\, b$, with coefficients encoded by the dual annular matrices S_x. The associativity property $(x\, a)\, b = x\, (a\, b)$ imposes the dual annular matrices to satisfy the quantum symmetry algebra (see Eq.(7)). In some simples cases, G itself appears as a subgraph of the Ocneanu graph, in other cases it appears as a module over the algebra of a particular subgraph. The methods we have described allow for the determination of the graph G even when orbifold and conjugation arguments from the $\mathcal{A}_k$ graphs do not apply (the exceptional cases). It can be used for a general affine algebra $\widehat{g}_k$ at any given level k, once the corresponding modular invariant is known.

In the next section, we present and illustrate this method by using several exceptional examples. In the $su(3)$ family, there are three exceptional graphs with self fusion. They are called $\mathcal{E}_5, \mathcal{E}_9$ and $\mathcal{E}_{21}$. In this paper we have chosen $\mathcal{E}_5$ (a kind of generalization of the E_6 case of $su(2)$, also treated as a standard example) and $\mathcal{E}_9$. The case of $\mathcal{E}_{21}$ (a kind of generalization of the E_8 case of $su(2)$) is actually very simple to discuss, even simpler than

$\mathcal{E}_5$ because it does not admit any non trivial module graph, and we could have described it as well, along the same lines. Results concerning $\mathcal{E}_{21}$ and its quantum symmetries can be found in [6, 25] (in those references, the graph itself is a priori given). The $su(3)$ - analogue of the E_7 case of $su(2)$, which is an exceptional twist of $\mathcal{D}_9$, can also be analysed thank's to the modular splitting formula, of course, but the discussion is quite involved (see [16, 15]). We refer to [26] for a description of an $\widehat{su}(4)$ example. In [8], these methods were applied to a non simply-laced example of the $su(2)$ family, where the initial partition function is not modular invariant (it is invariant under a particular congruence subgroup) and where there is no associated quantum groupoïd.

3.5 Comments

All module, associativity and compatibility conditions described here between the different set of matrices follow from properties of the quantum groupoïd $\mathcal{B}(G)$ constructed from the higher ADE graph G [18, 23, 25]. General results have been published on this quantum groupoïd (see [18, 7, 10, 17, 21]). But we are not aware of any definite list of properties that the graphs G should satisfy to obtain the right classification. The strategy adopted here is to take as granted the existence of a quantum groupoïd and its corresponding set of properties, and to derive the graph G as a by-product of the calculations, starting from the only knowledge of the modular invariant. Notice that this seems to be the method adopted by Ocneanu in order to produce his list of $SU(3)$ and $SU(4)$ graphs presented in [20]. One crucial check for the existence of the underlying quantum groupoïd is the existence of dimensional rules:

$$\dim(\mathcal{B}(G)) = \sum_{\lambda} d_{\lambda}^2 = \sum_{x} d_x^2 \ , \tag{24}$$

where the dimensions d_λ and d_x are calculated from the annular and dual annular matrices: $d_\lambda = \sum_{a,b}(F_\lambda)_{ab}$, $d_x = \sum_{a,b}(S_x)_{ab}$.

The method described in this article allows for the determination of the set of matrices $\{F_\lambda, N_\lambda, O_x, V_{\lambda\mu}, S_x\}$ and the corresponding graphs. Once the ADE Coxeter graph G has been obtained, and following the works of [4, 5, 25], we can also propose a realization of its quantum symmetry algebra as a particular tensor product of graph algebras. With this realization at hand, the matrices O_x and S_x can be expressed in a much more economic way.

4 Examples

4.1 The E_6 case of $\widehat{su}(2)$

We start with the $\widehat{su}(2)_{10}$ modular invariant partition function:

$$\mathcal{Z} = |\chi_0 + \chi_6|^2 + |\chi_3 + \chi_7|^2 + |\chi_4 + \chi_{10}|^2 \ , \tag{25}$$

where χ_λ's are the characters of $\widehat{su}(2)_{10}$ with $0 \leq \lambda \leq 10$. The total number of irreps is $d_{\mathcal{A}} = 11$. The modular invariant matrix $\mathcal{M}$ is read from $\mathcal{Z}$ when the later is written $\mathcal{Z} = \sum_\lambda \chi_\lambda \, \mathcal{M}_{\lambda\mu} \, \bar{\chi}_\mu$. The fusion matrices are given by the truncated recurrence formulae of SU(2) irreps at level $k = 10$:

$$N_0 = \mathbb{1}_{11} \quad ; \quad N_1 = Ad(\mathcal{A}_{10}) \quad ; \quad N_{\lambda+1} = N_\lambda . N_1 - N_{\lambda-1} \tag{26}$$

where $Ad(\mathcal{A}_{10})$ is the adjacency matrix of the graph $\mathcal{A}_{10}$ which is the truncated Weyl alcove of SU(2) irreps at level $k = 10$ (usually called the Dynkin diagram A_{11}).

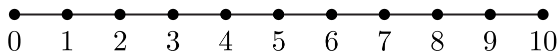

Figure 1: The $\mathcal{A}_{10} = A_{11}$ graph.

Determination of toric matrices W_z. Here we repeat the discussion given in [8]. We have $Tr(\mathcal{M}\mathcal{M}^\dagger) = 12$ which gives the dimension of the algebra of quantum symmetries $d_O = 12$. From the knowledge of $\mathcal{M}$ and the fusion matrices N_λ we determine all matrices $K_{\lambda\mu} = N_\lambda \, \mathcal{M} \, N_\mu^{tr}$. We calculate the rank of this family and find $r = 12$. Since $r = d_0$, the toric matrices W_{x0} are linearly independent and form a special basis of this vector space. For each matrix $K_{\lambda\mu}$ we look at their squared norm given by $||K_{\lambda\mu}||^2 = (K_{\lambda\mu})_{\lambda\mu}$ (conjugation is trivial for $SU(2)$ systems, $\lambda^* = \lambda$).

- For squared norm 1 we have 11 linearly independent matrices $K_{\lambda\mu}$, for example

$$K_{0,0} \quad K_{0,1} \quad K_{0,2} \quad K_{0,9} \quad K_{0,10} \quad K_{1,0} \quad K_{1,1} \quad K_{1,2} \quad K_{1,9} \quad K_{1,10} \quad K_{2,0} \, . \qquad (27)$$

  Each of these matrices define a toric matrix $W_x = W_{x0}$, so we get 11 out of the 12 toric matrices.

- For squared norm 2 we have 19 linearly independent matrices $K_{\lambda\mu}$, for example

$$\begin{array}{cccccccccc} K_{0,3} & K_{0,4} & K_{0,5} & K_{0,6} & K_{0,7} & K_{1,3} & K_{1,4} & K_{1,5} & K_{1,6} & K_{1,7} \\ K_{3,0} & K_{3,1} & K_{3,9} & K_{3,10} & K_{4,0} & K_{4,1} & K_{4,9} & K_{4,10} & K_{5,0} & \end{array} \qquad (28)$$

  In this list there are four matrices which are not equal to the sum of two already determined toric matrices, one of them being for instance $K_{0,3}$. They are therefore equal to the sum of an already determined toric matrix and the last one to be determined. We build the set of matrices $K_{0,3} - W_x$, where W_x runs in the list of determined toric matrices, and search for those which have non-negative integer coefficients. There is only one solution (namely $W_x = K_{1,9}$), and we get the last toric matrix as $K_{0,3} - K_{1,9}$.

- We have therefore determined the set of 12 toric matrices W_x, with $0 \leq x \leq 11$ and we can check our result by an explicit verification of the modular splitting equation (9).

Determination of $V_{\lambda\mu}$ Having determined the set of toric matrices W_x, we compute the set of matrices $K^x_{\lambda\mu} = N_\lambda \, W_x \, N_\mu^{tr}$. For $SU(2)$, all double fusion matrices $V_{\lambda\mu}$ are generated by the two fundamental matrices V_{10} and V_{01}. It is therefore sufficient to calculate the decomposition of K^x_{10} and K^x_{01} on the set of toric matrices from Eq.(16) to determine V_{10} and V_{01}. The calculation is straightforward, and choosing a special ordering in the set of indices x the resulting matrices are given by

$$V_{10} = \begin{pmatrix}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot
\end{pmatrix} \qquad V_{01} = \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot
\end{pmatrix}$$

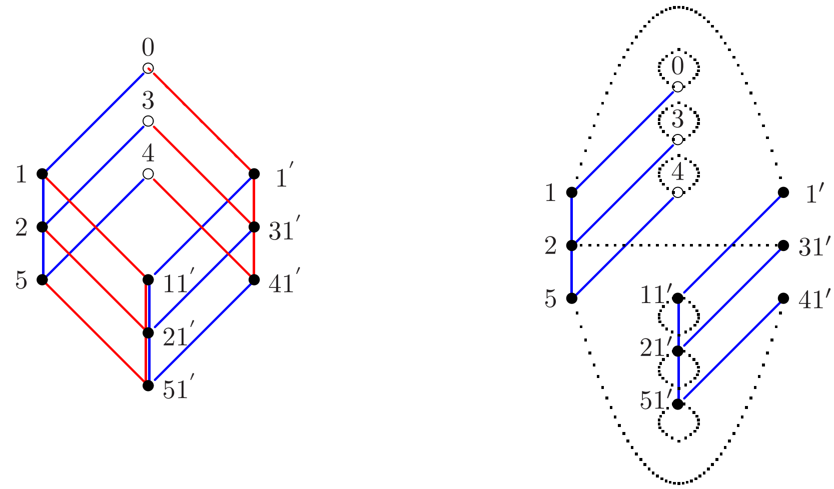

Figure 2: The E_6 Ocneanu graph displayed in two alternative ways.

The Ocneanu graph of quantum symmetries The two matrices V_{10} and V_{01} are the adjacency matrices of the graph of quantum symmetries (Ocneanu graph) associated to the initial modular invariant. The graph is displayed in figure 2, where the ordering for the labels[2] of the set of indices x is $\{0, 1, 2, 5, 4, 3\,,\ 1', 11', 21', 51', 41', 31'\}$. $O_1 = V_{10}$ is associated to the vertex 1 and encodes the multiplication by this vertex; the corresponding lines are displayed in blue. $O_{1'} = V_{01}$ is associated to the vertex $1'$ and encodes the multiplication by this vertex; the corresponding lines are displayed in red. Vertices 1 and $1'$ are called left and right chiral generators. The multiplication can be extended to the whole vector space spanned by the vertices of the Ocneanu graph. We refer to [25] for explicit expressions for the O_x matrices. There is a chiral conjugation on the graph that permutes the two chiral generators. The chiral operator C satisfies $O_{1'} = C^{-1}\, O_1\, C$. Another way of displaying the Ocneanu graph is to draw only the graph of multiplication by one chiral generator, say 1, and to associate (for example using dashed lines) each vertex with its chiral conjugate. Multiplication of a vertex x by $1'$ is obtained as follows: we start with x, follow the dashed lines to find its chiral conjugate y, then use the multiplication of y by 1, and pull-back using the dashed lines to obtain the result. This alternative way of drawing the Ocneanu graph is displayed at the r.h.s. of figure 2.

The Dynkin diagram E_6 The Ocneanu graph of figure 2 is made of two copies of the Dynkin diagram E_6. One copy, labelled by vertices $\{0, 1, 2, 5, 4, 3\}$, is a subalgebra of the quantum symmetry algebra, the other one, labelled by $\{1', 11', 21', 51', 41', 31'\}$ is only a module. With the method of modular splitting, we see that we find the Dynkin diagram E_6 as a subgraph of the Ocneanu graph. We refer to [4, 25] for expressions of the S_x matrices. Notice that there is a multiplication defined on the vector space spanned by vertices of the E_6 diagram: one says that it is of subgroup type and that E_6 exists not only as a module

[2] We choose the same labelling for the vertices as in [4, 5, 25].

(over A_{11}) but as a graph algebra. The algebra of quantum symmetries can be realized as the tensor square of the E_6 graph algebra, but the tensor product has to be taken over the subalgebra J spanned by $\{0,3,4\}$: $Oc(E_6) = E_6 \otimes_J E_6$ (for more details see [4, 5, 25]).

4.2 The $\mathcal{E}_5$ case of $\widehat{su}(3)$

We start with the $\widehat{su}(3)_5$ modular invariant partition function:

$$\begin{aligned} \mathcal{Z} = \; &= |\chi^5_{(0,0)} + \chi^5_{(2,2)}|^2 + |\chi^5_{(0,2)} + \chi^5_{(3,2)}|^2 + |\chi^5_{(2,0)} + \chi^5_{(2,3)}|^2 && (29) \\ &+ |\chi^5_{(2,1)} + \chi^5_{(0,5)}|^2 + |\chi^5_{(3,0)} + \chi^5_{(0,3)}|^2 + |\chi^5_{(1,2)} + \chi^5_{(5,0)}|^2 \,, && (30) \end{aligned}$$

where χ^5_λ's are the characters of $\widehat{su}(3)_5$, labelled by $\lambda = (\lambda_1, \lambda_2)$ with $0 \leq \lambda_1, \lambda_2 \leq 5$, $\lambda_1 + \lambda_2 \leq 5$. The modular invariant matrix $\mathcal{M}$ is read from $\mathcal{Z}$ when the later is written[3] $\mathcal{Z} = \sum_\lambda \chi_\lambda \, \mathcal{M}_{\lambda\mu} \, \bar{\chi}_\mu$. The number of irreps λ is $d_\mathcal{A} = 21$. λ is also considered as a label taking values on the integers $\lambda \in \{0, 1, 2, \ldots, 20\}$, corresponding to the choice of a certain order on the set of pairs (λ_1, λ_2). $\lambda = 0 = (0,0)$ is the trivial representation and there are two fundamental irreps $(1,0)$ and $(0,1) = (1,0)^*$, where $(\lambda_1, \lambda_2)^* = (\lambda_2, \lambda_1)$ is the conjugated irrep. $N_{(1,0)}$ is the adjacency matrix of the oriented graph $\mathcal{A}_5$, which is the truncated Weyl alcove of SU(3) irreps at level $k = 5$ (see figure 3). The fusion matrix $N_{(0,1)}$ is the transposed matrix of $N_{(1,0)}$ and is the adjacency matrix of the same graph with reversed arrows. Once $N_{(1,0)}$ is known, the other fusion matrices can be obtained from the **truncated recursion formulae of $SU(3)$ irreps**, applied for increasing level up to k:

$$\begin{aligned} N_{(\lambda,\mu)} &= N_{(1,0)}\, N_{(\lambda-1,\mu)} - N_{(\lambda-1,\mu-1)} - N_{(\lambda-2,\mu+1)} && \text{if } \mu \neq 0 \\ N_{(\lambda,0)} &= N_{(1,0)}\, N_{(\lambda-1,0)} - N_{(\lambda-2,1)} && (31) \\ N_{(0,\lambda)} &= (N_{(\lambda,0)})^{tr} \end{aligned}$$

where it is understood that $N_{(\lambda,\mu)} = 0$ if $\lambda < 0$ or $\mu < 0$.

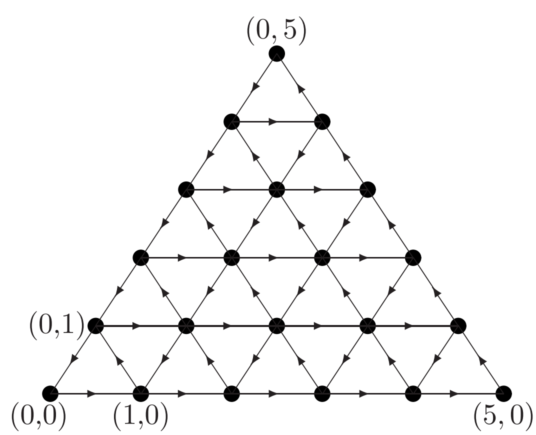

Figure 3: The $\mathcal{A}_5$ diagram.

[3] Some authors write instead $\mathcal{Z} = \sum_\lambda \chi_\lambda \, \mathcal{M}_{\lambda\mu^*} \, \bar{\chi}_\mu$, and therefore some care has to be taken in order to compare results since conjugated cases (in particular figures 4 and 5) must then be interchanged. Here we follow the convention made in [9].

Determination of toric matrices W_{z0} We have $d_O = Tr(\mathcal{M}\mathcal{M}^\dagger) = 24$. The matrices $K_{\lambda\mu} = N_\lambda\, \mathcal{M}\, N_\mu^{tr}$ span a vector space of dimension $r = 24$. This is therefore equal to d_O, the toric matrices W_{x0} are linearly independent and form a special basis for this vector space. For each matrix $K_{\lambda\mu}$ we calculate the squared norm given by $||K_{\lambda\mu}||^2 = (K_{\lambda\mu})_{\lambda^*\mu^*}$.

- For squared norm 1 we have 21 linearly independent matrices $K_{\lambda\mu}$, each one being equal to a toric matrix W_{z0}.
- There are 45 linearly independent matrices $K_{\lambda\mu}$ of squared norm 2. Some of them are equal to the sum of two already determined toric matrices. For a matrix not satisfying this property, say K_{ab}, we build the set of matrices $K_{ab} - W_x$, where W_x runs into the set of determined toric matrices, and look for those which have non-negative integer coefficients. This condition is strong enough and leads to only one solution (if K_{ab} is the sum of a determined matrix and a new one). We determine in that way the last three toric matrices.
- We have therefore determined the set of 24 toric matrices W_x, with $0 \leq x \leq 23$ and we can check our result by an explicit verification of the modular splitting equation (9).

Determination of $V_{\lambda\mu}$ Having determined the set of toric matrices W_{x0}, we compute the set of matrices $K^x_{\lambda\mu} = N_\lambda\, W_{x0}\, N^{tr}_\mu$. For $SU(3)$ cases, all double fusion matrices $V_{\lambda\mu}$ are generated by the two fundamental matrices $V_{(1,0),(0,0)}$, $V_{(0,0),(1,0)}$ and their transposed $V_{(0,1),(0,0)} = V^{tr}_{(1,0),(0,0)}$, $V_{(0,0),(0,1)} = V^{tr}_{(0,0),(1,0)}$. In order to determine these matrices, it is therefore sufficient to compute the decomposition of $K^x_{(1,0),(0,0)}$ and $K^x_{(0,0),(1,0)}$ on the set of toric matrices W_{x0} using Eq.(16). The calculation is straightforward. From the knowledge of the fundamental matrices $V_{(1,0),(0,0)}$, $V_{(0,0),(1,0)}$ and their transposed, all double fusion matrices $V_{\lambda\mu}$ are recursively calculated from Eqs.(17–19).

The Ocneanu graph of quantum symmetries The four fundamental matrices explicitly given below, in Eqs.(33), are the adjacency matrices of the graph of quantum symmetries (Ocneanu graph) associated to the initial modular invariant. We display in figure 4 the graph corresponding to the matrix $V_{(1,0),(0,0)}$ associated to the vertex labelled by $2_1 \otimes 1_0$. $V_{(0,0),(1,0)}$ is associated to the vertex $1_5 \otimes 2_0$, and instead of displaying the corresponding arrows, we display the action of the chiral conjugation C in order to not clutter the figure (warning: see the last footnote). The arrows corresponding to the matrix $V_{(0,1),(0,0)}$, associated to the vertex $2_2 \otimes 1_0$, are obtained by reversing the ones of figure 4; for the matrix $V_{(0,0),(0,1)}$, associated to the vertex $1_4 \otimes 2_0$, we use the chiral conjugation and the reversed arrows.

The generalized Dynkin diagram $\mathcal{E}_5$ The graph of figure 4 is made of two copies of the generalized Dynkin diagram $\mathcal{E}_5$. The $\mathcal{E}_5$ graph has 12 vertices denoted by $1_i, 2_i$, $i = 0, 1, \ldots, 5$. The unit is 1_0 and the generators are 2_1 and 2_2, the orientation of the graph corresponds to multiplication by 2_1. Conjugation corresponds to the symmetry with respect to the axis passing through vertices 1_0 and 1_3: $1_0^* = 1_0, 1_1^* = 1_5, 1_2^* = 1_4, 1_3^* = 1_3$; $2_0^* = 2_3, 2_1^* = 2_2, 2_4^* = 2_5$. The $\mathcal{E}_5$ graph determines in a unique way its graph algebra (it is a subgroup graph). The commutative multiplication table is given by:

$$\begin{array}{rcl} 1_i.1_j &=& 1_{i+j} \\ 1_i.2_j = 2_i.1_j &=& 2_{i+j} \\ 2_i.2_j &=& 2_{i+j} + 2_{i+j-3} + 1_{i+j-3} \end{array} \qquad i,j = 0,1,\ldots,5 \quad \mathrm{mod}\ 6 \qquad (32)$$

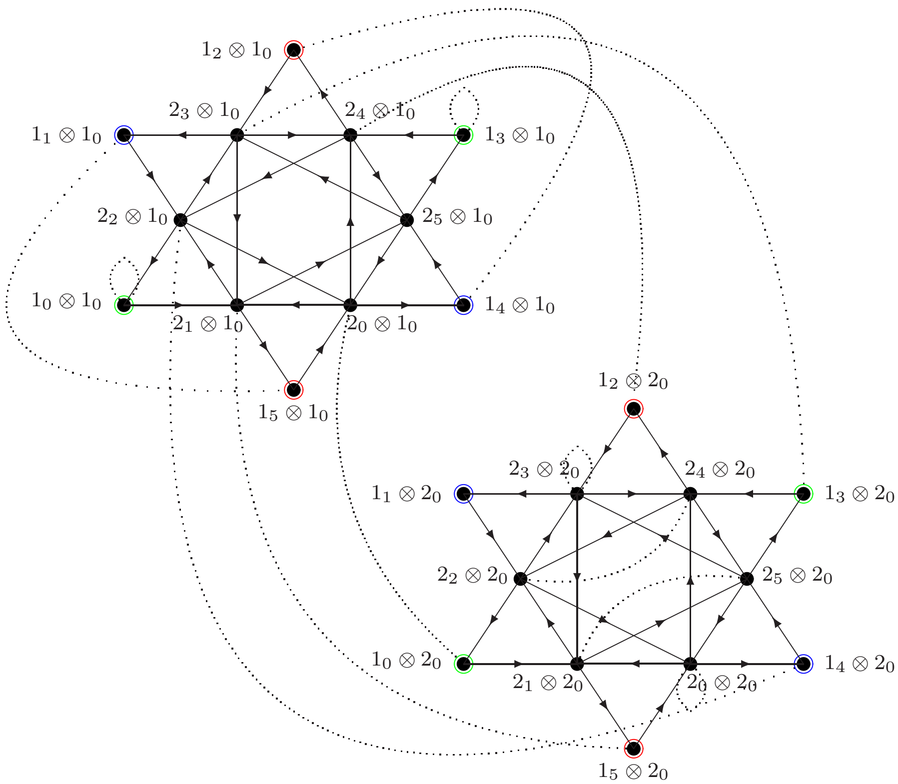

Figure 4: Ocneanu graph $Oc(\mathcal{E}_5)$. The two left chiral generators are $2_1 \otimes 1_0$ and $2_2 \otimes 1_0$, the two right chiral generators are $1_5 \otimes 2_0$ and $1_4 \otimes 2_0$.

From this multiplication table we get the graph algebra matrices G_a associated to the vertices $a \in \mathcal{E}_5$. The one corresponding to the generator 2_1 is the adjacency matrix of the graph. The vector space spanned by vertices of $\mathcal{E}_5$ is a module under the action of vertices of $\mathcal{A}_5$, the action being encoded by the annular matrices F_λ obtained form the recurrence relation (31) with starting point $F_{(0,0)} = \mathbb{1}_{12}$, $F_{(1,0)} = G_{2_1}$ and $F_{(0,1)} = G_{2_2}$.

Choosing a special ordering in the set of indices z of the algebra of quantum symmetries, and using the 12×12 graph algebra matrices G_a of the graph $\mathcal{E}_5$, the fundamental double fusion matrices are given by

$$V_{(1,0),(0,0)} = \left(\begin{array}{c|c} G_{2_1} & . \\ \hline . & G_{2_1} \end{array}\right) \qquad V_{(0,0),(1,0)} = \left(\begin{array}{c|c} . & G_{1_5} \\ \hline G_{1_2} & G_{1_2} + G_{1_5} \end{array}\right)$$
$$V_{(0,1),(0,0)} = \left(\begin{array}{c|c} G_{2_2} & . \\ \hline . & G_{2_2} \end{array}\right) \qquad V_{(0,0),(0,1)} = \left(\begin{array}{c|c} . & G_{1_4} \\ \hline G_{1_1} & G_{1_1} + G_{1_4} \end{array}\right) \tag{33}$$

Realization of $Oc(\mathcal{E}_5)$ The algebra of quantum symmetries $Oc(\mathcal{E}_5)$ can be realized as

$$Oc(\mathcal{E}_5) = \mathcal{E}_5 \otimes_J \mathcal{E}_5 \qquad \text{with } a \otimes_J b.c = a.b^* \otimes_J c \qquad \text{for } b \in J = \{1_i\}\ , \tag{34}$$

where J is a subalgebra characterized by modular properties (see [6, 25]). The algebra $Oc(\mathcal{E}_5)$ has dimension $12 \times 2 = 24$, and a basis is given by elements $a \otimes_J 1_0$ and $a \otimes_J 2_0$. The identifications in $Oc(\mathcal{E}_5)$ are given by:

$$\begin{array}{rcl} 1_i \otimes_J 1_j & = & 1_{i+j^*} \otimes_J 1_0 \\ 2_i \otimes_J 1_j & = & 2_{i+j^*} \otimes_J 1_0 \\ 1_i \otimes_J 2_j = 1_i \otimes_J 1_j.2_0 & = & 1_{i+j^*} \otimes_J 2_0 \\ 2_i \otimes_J 2_j = 2_i \otimes_J 1_j.2_0 & = & 2_{i+j^*} \otimes_J 2_0 \end{array} \tag{35}$$

The chiral conjugation is defined by $(a \otimes_J b)^C = b \otimes_J a$. The left chiral generator is $2_1 \otimes_J 1_0$ and the right chiral generator is $1_0 \otimes_J 2_1 = 1_5 \otimes_J 2_0$. Multiplication in $Oc(\mathcal{E}_5)$ is defined from the multiplication (32) of $\mathcal{E}_5$ together with the identifications (35), and is encoded by the quantum symmetries matrices O_x. We get:

$$O_{x=a\otimes_J 1_0} = \begin{pmatrix} G_a & \cdot \\ \cdot & G_a \end{pmatrix} \qquad O_{x=a\otimes_J 2_0} = \begin{pmatrix} \cdot & G_a \\ G_a.G_{1_3} & G_a(\mathbb{1}+G_{1_3}) \end{pmatrix} \tag{36}$$

The vector space of $\mathcal{E}_5$ vertices is also a module under the action of vertices of $Oc(\mathcal{E}_5)$ defined by $(a \otimes_J 1_0).b = a.b$ and $(a \otimes_J 2_0).b = a.2_0.b$. The dual annular matrices S_x are given by $S_{x=a\otimes_J 1_0} = G_a$ and $S_{x=a\otimes_J 2_0} = G_{2_0}.G_a$. We check the dimensional rules $\dim(\mathcal{B}(\mathcal{E}_5)) = \sum_\lambda d_\lambda^2 = \sum_x d_x^2 = 29\,376$.

4.3 The $\mathcal{E}_5^*$ case of $\widehat{su}(3)$

We start now with the following $\widehat{su}(3)_5$ modular invariant partition function:

$$\begin{array}{rcl} \mathcal{Z} & = & |\chi^5_{(0,0)} + \chi^5_{(2,2)}|^2 + |\chi^5_{(3,0)} + \chi^5_{(0,3)}|^2 + [(\chi^5_{(0,2)} + \chi^5_{(3,2)}).(\overline{\chi^5_{(2,0)}} + \overline{\chi^5_{(2,3)}}) + \text{h.c.}] \\ & & +(\chi^5_{(2,1)} + \chi^5_{(0,5)}).(\overline{\chi^5_{(1,2)}} + \overline{\chi^5_{(5,0)}}) + \text{h.c.}]\ , \end{array} \tag{37}$$

and compute the modular matrix[4] $\mathcal{M}$. The fusion matrices N_λ are the same as in the previous case.

Determination of toric matrices and double fusion matrices We have $d_O = Tr(\mathcal{M}\mathcal{M}^\dagger) = 24$. The matrices $K_{\lambda\mu} = N_\lambda\,\mathcal{M}\,N_\mu^{tr}$ span a vector space of dimension $r = d_O = 24$. The discussion is the same as in the previous case.

- For squared norm 1 we have 21 linearly independent matrices $K_{\lambda\mu}$ defining 21 toric matrices W_{z0}.
- There are 45 linearly independent matrices $K_{\lambda\mu}$ of squared norm 2 and the last three toric matrices W_{z0} can be obtained.

Once the toric matrices have been determined, the double fusion matrices are obtained straightforwardly. For the fundamental ones we get:

$$\begin{array}{ll} V_{(1,0),(0,0)} = \left(\begin{array}{c|c} G_{2_1} & \cdot \\ \hline \cdot & G_{2_1} \end{array}\right) & V_{(0,0),(1,0)} = \left(\begin{array}{c|c} \cdot & G_{1_1} \\ \hline G_{1_4} & G_{1_1}+G_{1_4} \end{array}\right) \\ V_{(0,1),(0,0)} = \left(\begin{array}{c|c} G_{2_2} & \cdot \\ \hline \cdot & G_{2_2} \end{array}\right) & V_{(0,0),(0,1)} = \left(\begin{array}{c|c} \cdot & G_{1_2} \\ \hline G_{1_5} & G_{1_2}+G_{1_5} \end{array}\right) \end{array} \tag{38}$$

[4]Same remark as in the last footnote.

The Ocneanu graph of quantum symmetries We display in figure 5 the graph corresponding to the matrix $V_{(1,0),(0,0)}$ associated with the vertex labelled by $2_1 \otimes 1_0$. $V_{(0,0),(1,0)}$ is associated with the vertex $1_1 \otimes 2_0$. The algebra of quantum symmetries can be realized as

$$Oc(\mathcal{E}_5^*) = \mathcal{E}_5 \otimes_J \mathcal{E}_5 \qquad \text{with } a \otimes_J b.c = a.b \otimes_J c \qquad \text{for } b \in J = \{1_i\} \,. \tag{39}$$

The algebra $Oc(\mathcal{E}_5^*)$ has also dimension $12 \times 2 = 24$ and a basis is given by elements $a \otimes_J 1_0$ and $a \otimes_J 2_0$. The identifications in $Oc(\mathcal{E}_5^*)$ are given by (different from those of $Oc(\mathcal{E}_5)$)

$$\begin{array}{rcl} 1_i \otimes_J 1_j & = & 1_{i+j} \otimes_J 1_0 \\ 2_i \otimes_J 1_j & = & 2_{i+j} \otimes_J 1_0 \\ 1_i \otimes_J 2_j = 1_i \otimes_J 1_j.2_0 & = & 1_{i+j} \otimes_J 2_0 \\ 2_i \otimes_J 2_j = 2_i \otimes_J 1_j.2_0 & = & 2_{i+j} \otimes_J 2_0 \end{array} \tag{40}$$

The left chiral generator is $2_1 \otimes_J 1_0$ and the right chiral generator is $1_0 \otimes_J 2_1 = 1_1 \otimes_J 2_0$. The algebra $Oc(\mathcal{E}_5^*)$ is isomorphic to $Oc(\mathcal{E}_5)$, the quantum symmetry matrices O_x are still given by (36). The difference is in the chiral conjugacy.

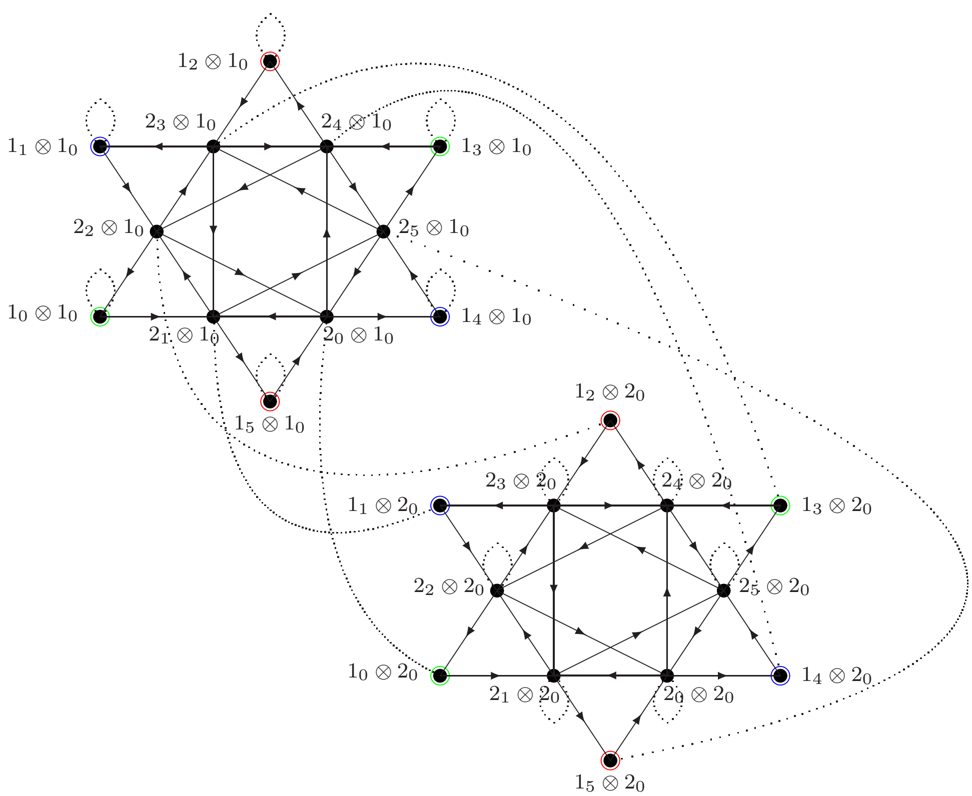

Figure 5: Ocneanu graph $Oc(\mathcal{E}_5)^*$. The two left chiral generators are $2_1 \otimes 1_0$ and $2_2 \otimes 1_0$, the two right chiral generators are $1_1 \otimes 2_0$ and $1_2 \otimes 2_0$.

The generalized Dynkin diagram $\mathcal{E}_5^* = \mathcal{E}_5/3$ The graph associated to the initial modular invariant (37) is a module graph for the Ocneanu graph displayed on figure 5. It must therefore be a module graph of the $\mathcal{E}_5$ graph itself: it is obtained as the Z_3-orbifold graph of $\mathcal{E}_5$ (see

[14]). We write this module property $a\,\tilde{b} = \sum_{\tilde{c}} (F_a^{\mathcal{E}})_{\tilde{b}\tilde{c}}\,\tilde{c}$, for $a \in \mathcal{E}_5$ and $\tilde{b}, \tilde{c} \in \mathcal{E}_5/3$, encoded by the 12 matrices $F_a^{\mathcal{E}}$. From the associative property $(a.b).\tilde{c} = a.(b.\tilde{c})$, these matrices must satisfy the same commutation relations (32) as the graph algebra of $\mathcal{E}_5$, and can be recursively calculated from $F_{2_1}^{\mathcal{E}}$, which is the adjacency matrix of the $\mathcal{E}_5/3$ graph displayed on figure 6. The $\mathcal{E}_5/3$ graph is also a module over the algebra of quantum symmetries, the action being defined by $(a \otimes_J 1_0).\tilde{b} = a.\tilde{b}$ and $(a \otimes_J 2_0).b = a.2_0.\tilde{b}$. The dual annular matrices S_x are therefore given by $S_{x=a\otimes_J 1_0} = F_a^{\mathcal{E}}$ and $S_{x=a\otimes_J 2_0} = F_{2_0}^{\mathcal{E}}.F_a^{\mathcal{E}}$. We check the dimensional rules $\dim(\mathcal{B}(\mathcal{E}_5^*)) = \sum_\lambda d_\lambda^2 = \sum_x d_x^2 = 3\,264$.

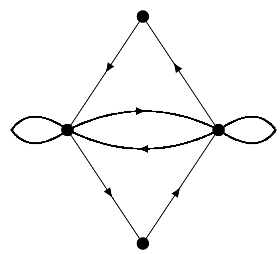

Figure 6: The $\mathcal{E}_5^* = \mathcal{E}_5/3$ generalized Dynkin diagram.

So both graphs $G = \mathcal{E}_5$ and $\mathcal{E}_5/3$ have the same (isomorphic) algebra $Oc(G)$ of quantum symmetries, but its realization in terms of tensor square of $\mathcal{E}_5$ is different in the two cases, as well as the chiral conjugation, and, of course, its module action on $\mathcal{E}_5$ or on $\mathcal{E}_5/3$.

4.4 The $\mathcal{E}_9$ case of $su(3)$

We start with the following $\widehat{su}(3)_9$ modular invariant partition function:

$$\mathcal{Z} = |\chi_{0,0}^9 + \chi_{0,9}^9 + \chi_{9,0}^9 + \chi_{1,4}^9 + \chi_{4,1}^9 + \chi_{4,4}^9|^2 + 2\,|\chi_{2,2}^9 + \chi_{2,5}^9 + \chi_{5,2}^9|^2\,, \tag{41}$$

where χ_λ^9's are the characters of $\widehat{su}(3)_9$, labelled by $\lambda = (\lambda_1, \lambda_2)$ with $0 \leq \lambda_1, \lambda_2 \leq 9$, $\lambda_1 + \lambda_2 \leq 9$. The modular invariant matrix is recovered from $\mathcal{Z} = \sum_\lambda \chi_\lambda \mathcal{M}_{\lambda\mu} \overline{\chi}_\mu$. The number of irreps is $d_{\mathcal{A}} = 55$. The fusion matrix $N_{(1,0)}$ is the adjacency matrix of the $\mathcal{A}_9$ graph, the truncated Weyl alcove of SU(3) irreps at level 9. The other fusion matrices are determined by the recurrence relation (32).

Determination of toric matrices W_{z0} We have $d_O = Tr(\mathcal{M}\mathcal{M}^\dagger) = 72$ and therefore an Ocneanu algebra with 72 generators z and also 72 toric matrices W_{z0}. However these toric matrices span a vector space of dimension $r = 45 < 72$, i.e. they are not all linearly independent. For each matrix $K_{\lambda\mu} = N_\lambda \mathcal{M} N_\mu^{tr}$ we consider its "squared norm" defined by $||K_{\lambda\mu}||^2 = (K_{\lambda\mu})_{\lambda^*\mu^*}$:

- There are 27 matrices $K_{\lambda\mu}$ with squared norm 1, each one defines a toric matrix W_{z0}.
- There are 12 linearly independent matrices $K_{\lambda\mu}$ with squared norm 2, but each one is equal to the sum of two already determined matrices. We don't find any new toric matrix in this family.
- There are 21 linearly independent matrices $K_{\lambda\mu}$ of squared norm 3, none of them being equal to the sum of three already obtained matrices. Twelve amoung these 21 are equal

to the sum of one determined matrix and a matrix having coefficients multiple of 2. A solution leading to squared norm 3 is to define a new toric matrix by dividing by 2 the matrix with coefficients multiple of 2, and adding them to the list with a multiplicity two. From these twelve we obtain actually only eight different toric matrices (because some are obtained more than once), each one coming with multiplicity two. Nine of the 21 matrices have coefficients which are multiple of 3. We define nine new toric matrices by dividing these matrices by 3, each toric matrix obtained in that way appearing with multiplicity 3. At that stage, we have determined $27 + (2 \times 8) + (3 \times 9) = 70$ toric matrices.

- There are 24 linearly independent matrices $K_{\lambda\mu}$ with squared norm 4, but each one is equal to the sum of four already obtained matrices. We don't recover any new toric matrix. This is also the case for squared norm 5.
- There are 10 linearly independent matrices $K_{\lambda\mu}$ with squared norm 6. We discard those that can be written as a linear combination of already determined toric matrices, and pick up one of the others, for example K_{ab}. We build the list of matrices $K_{ab} - W_x$, for W_x running into the set of already obtained toric matrices, searching for matrices with non-negative coefficients. With our choice, it is so that K_{ab} is the sum of two times a toric matrix plus a new one which has matrix elements multiple of 2. Dividing the later by 2 and adding it to the list, with multiplicity 2, we get the last toric matrices.

We have indeed therefore determined the 72 toric matrices, 45 (=27+9+8+1) of them being linearly independent, but appearing with multiplicities (27 of multiplicity one, 9 (=8+1) of multiplicity two and 9 of multiplicity three). We can check the result by a direct substitution in the $55 \times 55 = 3025$ matrix equations over non-negative integers (12).

Determination of $V_{(1,0),(0,0)}$ and $V_{(0,0),(1,0)}$ We compute the set of matrices $K^x_{\lambda\mu} = N_\lambda W_{x0} N^{tr}_\mu$ for $\{\lambda\mu\} = \{(1,0),(0,0)\}$ and $\{(0,0),(1,0)\}$, and decompose them on the family (not a base) of toric matrices W_{z0} using (12). Since the W_{z0} are not linearly independent, the decomposition is not unique, and we introduce some undetermined coefficients. Imposing that they should be non-negative integers allows to fix some of them or to obtain relations between them. More constraints come from the fact that we have $V_{(0,0),(1,0)} = C.V_{(0,0),(1,0)}.C^{-1}$, where C is the chiral operator. Notice that C itself is deduced from the previous relation even if $V_{(0,0),(1,0)}$ and $V_{(0,0),(1,0)}$ still contain free parameters, by using the fact that it is a permutation matrix. Choosing an appropriate order on the set of indices z, we obtain the following structure for $V_{(1,0),(0,0)}$:

$$V_{(1,0),(0,0)} = \begin{pmatrix} Ad(\mathcal{E}_9) & . & . & . & . & . \\ . & Ad(\mathcal{E}_9) & . & . & . & . \\ . & . & Ad(\mathcal{E}_9) & . & . & . \\ . & . & . & Ad(\mathcal{M}_9) & . & . \\ . & . & . & . & Ad(\mathcal{M}_9) & . \\ . & . & . & . & . & Ad(\mathcal{M}_9) \end{pmatrix} \tag{42}$$

where $Ad(\mathcal{E}_9)$ and $Ad(\mathcal{M}_9)$ are 12×12 matrices (still containing some unknown coefficients).

The generalized Dynkin diagram $\mathcal{E}_9$ The $Ad(\mathcal{E}_9)$ matrix is the adjacency matrix of the graph $\mathcal{E}_9$ displayed on the l.h.s. of figure 7. It possesses a $\mathbb{Z}_3$-symmetry corresponding to the permutation of the three "wings" formed by vertices 0_i, 1_i and 2_i. The undetermined coefficients of the adjacency matrix reflect this symmetry; they are simply fixed once an ordering has been chosen for the vertices (something similar happens for the D_{even} series of the $su(2)$ family).

The vector space of the $\mathcal{E}_9$ graph is a module over the left-right action of the graph algebra of the $\mathcal{A}_9$ graph, encoded by the annular matrices $F^{\mathcal{E}}_\lambda$

$$\mathcal{A}_9 \times \mathcal{E}_9 \to \mathcal{E}_9 : \quad \lambda \cdot a = a \cdot \lambda = \sum_b (F^{\mathcal{E}}_\lambda)_{ab}\, b \qquad \lambda \in \mathcal{A}_9\ , \quad a, b \in \mathcal{E}_9\ . \tag{43}$$

The $F^{\mathcal{E}}_\lambda$ matrices give a representation of dimension 12 of the fusion algebra and are determined from the recursion relation (32) with $F^{\mathcal{E}}_{(0,0)} = \mathbb{1}_{12\times 12}$, $F^{\mathcal{E}}_{(1,0)} = Ad(\mathcal{E}_9)$. We notice that fundamental matrices (for instance $F_{(1,0)}$) contain, in this case, elements bigger than 1, however, the "rigidity[5] condition" $(F_\lambda)_{ab} = (F_{\lambda^*})_{ba}$ holds, so that this example is indeed an higher analogue of the ADE graphs, not an higher analogue of the non simply laced cases. Triality and conjugation compatible with the action of $\mathcal{A}_9$ can be defined on the $\mathcal{E}_9$ graph. Triality is denoted by the index $i \in \{0, 1, 2\}$ in the set of vertices $0_i, 1_i, 2_i$. The conjugation corresponds to the vertical axis going through vertices 0_0 and 3_0: $0_0^* = 0_0, 1_0^* = 2_0, 3_0^* = 3_0$, $0_1^* = 0_2, 1_1^* = 2_2, 1_2^* = 2_1, 3_1^* = 3_2$. The $\mathbb{Z}_3$-symmetry action on vertices of $\mathcal{E}_9$ is denoted ρ_3. The axis formed by vertices 3_i is invariant under ρ_3 and the symmetry permutes the three wings $\rho_3(0_0) = 1_0$, $\rho_3(1_0) = 2_0$, $\rho_3(2_0) = 0_0$; $\rho_3(0_1) = 1_1$, $\rho_3(1_1) = 2_1$, $\rho_3(2_1) = 0_1$; $\rho_3(0_2) = 1_2$, $\rho_3(1_2) = 2_2$, $\rho_3(2_2) = 0_2$. Once we have fixed the origin of the graph (the vertex 0_0), the graph still possesses a $\mathbb{Z}_2$-symmetry corresponding to the permutation of the two remaining wings, formed by vertices 1_i and 2_i. We denote ρ_2 this operation: $\rho_2(1_i) = 2_i$ and $\rho_2^2 = \mathbb{1}$.

The $\mathcal{E}_9$ graph has also self-fusion: the vector space spanned by its vertices has an associative algebra structure, with non-negative structure constants, compatible with the action of $\mathcal{A}_9$. 0_0 is the unity and the two conjugated generators are 0_1 and 0_2. The graph itself is also the Cayley graph of multiplication by 0_1. Due to the symmetry of the wings of the graph, the knowledge of the multiplication by generators 0_1 and 0_2 is not sufficient to reconstruct the whole multiplication table; we have to impose structure coefficients to be non-negative integers in order to determine a unique solution (see [6, 25]). The whole multiplication table is encoded in the graph algebra matrices G_a, for $a \in \mathcal{E}_9$. We give below the expression for the matrices G_{1_0} and G_{2_0}, the other matrices are defined by $G_{0_0} = \mathbb{1}$, $G_{0_1} = G_{0_2}^{tr} = Ad(\mathcal{E}_9)$, $G_{3_0} = G_{0_1}\,G_{0_2} - G_{0_0}$, $G_{3_2} = G_{3_1}^{tr} = G_{0_1}\,G_{0_1} - G_{0_2}$, $G_{1_1} = G_{2_2}^{tr} = G_{0_1}\,G_{1_0}$, $G_{1_2} = G_{2_1}^{tr} = G_{0_2}\,G_{1_0}$. In the ordered basis $(0_0, 1_0, 2_0, 3_0; 0_1, 1_1, 2_1, 3_1; 0_2, 1_2, 2_2, 3_2)$, G_{1_0} and

[5] We call it that way because of its relation with the theory of rigid categories, see for instance [21]).

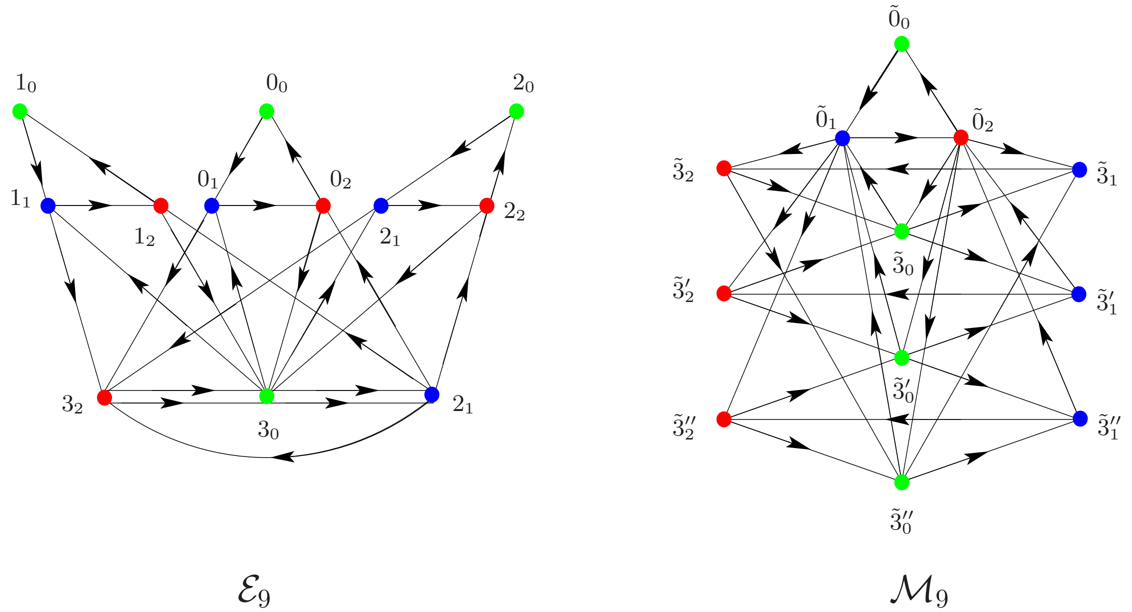

Figure 7: The graphs $\mathcal{E}_9$ and $\mathcal{M}_9$

G_{2_0} are given by

$$G_{1_0} = G_{2_0}^T = \begin{pmatrix}
. & 1 & . & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . \\
1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix} \tag{44}$$

Notice that multiplication by 1_0 corresponds to the $\mathbb{Z}_3$ operation: $1_0.a = \rho_3(a)$. The matrix G_{1_0} is the permutation matrix representing the action of the $\mathbb{Z}_3$ operator ρ_3: $(G_{1_0})_{ab} = \delta_{b,\rho_3(a)}$. We have $(G_{1_0})^3 = \mathbb{1}$ and $(G_{1_0})^2 = G_{2_0}$, so G_{2_0} represents the operator $(\rho_3)^2$.

Other aspects and properties of the $\mathcal{E}_9$ graph and of its algebra of quantum symmetries (semi-simple structure of the associated quantum groupoïd, semi-simple structure of $Oc(\mathcal{E}_9)$ itself, quantum dimensions and quantum mass) are presented in [6, 25, 9].

The generalized Dynkin diagram $\mathcal{M}_9$ The matrix $Ad(\mathcal{M}_9)$ is a 12×12 matrix with some unknown coefficients to be determined. Imposing this matrix to be the adjacency matrix of a graph such that the vector space spanned by its vertices is a module over the graph algebras of $\mathcal{A}_9$ *and* of $\mathcal{E}_9$ leads to a unique solution. The graph is displayed on the r.h.s. of figure 7 and corresponds to the $\mathbb{Z}_3$-orbifold graph of $\mathcal{E}_9$, denoted $\mathcal{M}_9 = \mathcal{E}_9/3$.

The vector space spanned by vertices of the $\mathcal{M}_9$ graph is a module over the left-right action of the graph algebra of $\mathcal{A}_9$ encoded by the annular matrices $F_\lambda^{\mathcal{M}}$

$$\mathcal{A}_9 \times \mathcal{M}_9 \to \mathcal{M}_9 : \quad \lambda \cdot \tilde{a} = \tilde{a} \cdot \lambda = \sum_{\tilde{b}} (F_\lambda^{\mathcal{M}})_{\tilde{a}\tilde{b}} \, \tilde{b} \qquad\qquad \lambda \in \mathcal{A}_9 \; , \quad \tilde{a}, \tilde{b} \in \mathcal{M}_9 \; . \tag{45}$$

The $F_\lambda^{\mathcal{M}}$ matrices give a representation of dimension 12 of the fusion algebra and can be determined from the recursion relation (32) with $F_{(0,0)}^{\mathcal{M}} = \mathbb{1}_{12\times 12}$, $F_{(1,0)}^{\mathcal{M}} = Ad(\mathcal{M}_9)$. Triality and conjugation compatible with the action of $\mathcal{A}_9$ can be defined on the $\mathcal{M}_9$ graph. Triality is denoted by the index $i \in \{0,1,2\}$ in the set of vertices $\tilde{a}_i \in \mathcal{M}_9$. The conjugation corresponds to the vertical axis going through vertex $\tilde{0}_0$: $\tilde{0}_0^* = \tilde{0}_0, \tilde{0}_1^* = \tilde{0}_2, \tilde{3}_0^* = \tilde{3}_0$, $\tilde{3}_0'^{\,*} = \tilde{3}_0'$, $\tilde{3}_0''^{*} = \tilde{3}_0''$, $\tilde{3}_1^* = \tilde{3}_2, \tilde{3}_1'^{\,*} = \tilde{3}_2', \tilde{3}_1''^{*} = \tilde{3}_2''$.

The vector space spanned by vertices of $\mathcal{M}_9$ is also a module under the action of the graph algebra of $\mathcal{E}_9$. Here we will distinguish between left and right action. The left action of $\mathcal{E}_9$ is encoded by a set of 12×12 matrices denoted P_λ^ℓ

$$\mathcal{E}_9 \times \mathcal{M}_9 \to \mathcal{M}_9 : \quad a \cdot \tilde{b} = \sum_{\tilde{c}} (P_a^\ell)_{\tilde{b}\tilde{c}} \, \tilde{c} \qquad\qquad a \in \mathcal{E}_9 \; , \quad \tilde{b}, \tilde{c} \in \mathcal{M}_9 \; . \tag{46}$$

The right action of $\mathcal{E}_9$ is defined via the $\mathbb{Z}_2$ operator ρ_2:

$$\tilde{b} \cdot a = \rho_2(a) \cdot \tilde{b} \tag{47}$$

and is encoded by a set of 12×12 matrices denoted P_a^r: $\tilde{b} \cdot a = \sum_{\tilde{c}} (P_a^r)_{\tilde{b}\tilde{c}} \, \tilde{c}$ with $P_a^r = P_{\rho_2(a)}^\ell$. The module property $(a \cdot b) \cdot \tilde{c} = a \cdot (b \cdot \tilde{c})$ imposes P_a^ℓ matrices to form a representation of the graph algebra of $\mathcal{E}_9$; they satisfy $P_a^\ell \, P_b^\ell = \sum_c (G_a)_{bc} P_c^\ell$. Similar relations exist for P_a^r matrices. We can compute the set of matrices P_a^ℓ using the multiplicative structure of $\mathcal{E}_9$. We give below the expression for $P_{1_0}^\ell$ and $P_{2_0}^\ell$, the other matrices being defined by $P_{0_0}^\ell = \mathbb{1}$, $P_{0_1}^\ell = (P_{0_2}^\ell)^{tr} = Ad(\mathcal{M}_9)$, $P_{3_0}^\ell = P_{0_1}^\ell P_{0_2}^\ell - P_{0_0}^\ell$, $P_{3_2}^\ell = (P_{3_1}^\ell)^{tr} = P_{0_1}^\ell \, P_{0_1}^\ell - P_{0_2}^\ell$, $P_{1_1}^\ell = (P_{2_2}^\ell)^{tr} = P_{0_1}^\ell \, P_{1_0}^\ell$, $P_{1_2}^\ell = (P_{2_1}^\ell)^{tr} = P_{0_2}^\ell \, P_{1_0}^\ell$. In the ordered basis $(\tilde{0}_0, \tilde{3}_0, \tilde{3}_0', \tilde{3}_0''; \tilde{0}_1, \tilde{3}_1, \tilde{3}_1', \tilde{3}_1''; \tilde{0}_2, \tilde{3}_2, \tilde{3}_2', \tilde{3}_2'')$, $P_{1_0}^\ell$ and $P_{2_0}^\ell$ are given by

$$P_{1_0}^\ell = (P_{2_0}^\ell)^{tr} = \begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot
\end{pmatrix} \tag{48}$$

There is also an operator ρ_3' acting on vertices of the $\mathcal{M}_9$ graph, inherited from the $\mathbb{Z}_3$ symmetry of the $\mathcal{E}_9$ graph through the orbifold procedure. It satisfies the following property:

$$\rho_3(a) \, \tilde{b} = a \, \rho_3'(\tilde{b}) \tag{49}$$

We have $1_0\, a = \rho_3(a)$, so $\rho_3'(\tilde{a}) = 1_0\, \tilde{a}$. It is defined by $\rho_3'(\tilde{0}_i) = \tilde{0}_i$, $\rho_3'(\tilde{3}_i) = \tilde{3}_i'$, $\rho_3'(\tilde{3}_i') = \tilde{3}_i''$, $\rho_3'(\tilde{3}_i'') = \tilde{3}_i$, for $i = 0, 1, 2$. The matrix $P^\ell_{1_0}$ is therefore the permutation matrix representing the action of the $\mathbb{Z}_3$ operator ρ_3'. We have $(P^\ell_{1_0})^3 = \mathbb{1}$ and $(P^\ell_{1_0})^2 = P^\ell_{2_0}$, so $P^\ell_{2_0}$ represents the operator $(\rho_3')^2$.

The vector space $\mathcal{E}_9 \oplus \mathcal{M}_9$ We define the vector space $H = \mathcal{E}_9 \oplus \mathcal{M}_9$, and we want to define (this will be used later) an associative product on H with the following structure:

$\nearrow$	$\mathcal{E}_9$	$\mathcal{M}_9$
$\mathcal{E}_9$	$\mathcal{E}_9$	$\mathcal{M}_9$
$\mathcal{M}_9$	$\mathcal{M}_9$	$\mathcal{E}_9$

We define the following multiplications:

$$\begin{aligned} \mathcal{E}_9 \times \mathcal{E}_9 \to \mathcal{E}_9 \quad &: \quad a\, b = \sum_c (G_a)_{bc}\, c \\ \mathcal{E}_9 \times \mathcal{M}_9 \to \mathcal{M}_9 \quad &: \quad a\, \tilde{b} = \sum_{\tilde{c}} (P^\ell_a)_{\tilde{b}\tilde{c}}\, \tilde{c} \\ \mathcal{M}_9 \times \mathcal{E}_9 \to \mathcal{M}_9 \quad &: \quad \tilde{b}\, a = \sum_{\tilde{c}} (P^r_a)_{\tilde{b}\tilde{c}}\, \tilde{c} \\ \mathcal{M}_9 \times \mathcal{M}_9 \to \mathcal{E}_9 \quad &: \quad \tilde{a}\, \tilde{b} = \sum_c (H_{\tilde{a}})_{\tilde{b}c}\, c. \end{aligned} \tag{50}$$

The associativity property on H reads $a\,(b\,c) = (a\,b)\,c$; $a\,(b\,\tilde{c}) = (a\,b)\,\tilde{c}$; $a\,(\tilde{b}\,c) = (a\,\tilde{b})\,c$; $\tilde{a}\,(b\,c) = (\tilde{a}\,b)\,c$; $a\,(\tilde{b}\,\tilde{c}) = (a\,\tilde{b})\,\tilde{c}$; $\tilde{a}\,(b\,\tilde{c}) = (\tilde{a}\,b)\,\tilde{c}$; $\tilde{a}\,(\tilde{b}\,c) = (\tilde{a}\,\tilde{b})\,c$; $\tilde{a}\,(\tilde{b}\,\tilde{c}) = (\tilde{a}\,\tilde{b})\,\tilde{c}$, and induce a set of relations between matrices G_a, P^ℓ_a, P^r_a and H_a. In order to satisfy them, the unique solution for coefficients of the H_c matrices is given by:

$$(H_c)_{\tilde{a}\tilde{b}} = (P^\ell_{\rho_2(c)})_{\tilde{a}^*\tilde{b}} = (P^r_c)_{\tilde{a}^*\tilde{b}}\,. \tag{51}$$

The Ocneanu algebra of quantum symmetries and a realization The matrix $V_{(1,0),(0,0)}$ is the adjacency matrix of the left chiral part of the Ocneanu graph. The graph is composed of six subgraphs, three copies of the $\mathcal{E}_9$ graph and three copies of the $\mathcal{M}_9$ graph, as showed on figure 8. We label the vertices as follows: $x = a \otimes 0_i$ with $a, 0_i \in \mathcal{E}_9$ and $i = 0,\ 1,\ 2$ for vertices of $\mathcal{E}_9$-type subgraphs and $x = \tilde{a} \otimes \tilde{3}_i$ with $\tilde{a}, \tilde{3}_i \in \mathcal{M}_9$ and $i = 0,\ 1,\ 2$ for vertices of $\mathcal{M}_9$-type subgraphs. The matrix $V_{(1,0),(0,0)}$ corresponds to the multiplication by the left chiral generator $0_1 \otimes 0_0$. The matrix $V_{(0,0),(1,0)}$ is the adjacency matrix of the right chiral part of the Ocneanu graph $Oc(\mathcal{E}_9)$, and corresponds to the multiplication by the right chiral generator $0_0 \otimes 0_1$. The dashed lines in the graph corresponds to the chiral operator C. We have $V_{(0,0),(1,0)} = C V_{(1,0),(0,0)} C^{-1}$. The multiplication by $0_0 \otimes 0_1$ is obtained as follows. We start with x, apply C, multiply the result by $0_1 \otimes 0_0$, and apply $C^{-1} = C$. From matrices $V_{(1,0),(0,0)}$ and $V_{(0,0),(1,0)}$ all others $V_{\lambda\mu}$ (hence also the double toric matrices W_{xy}) are calculated straightforwardly using equations (17–19).

From the multiplication by chiral left and right generators $0_1 \otimes 0_0$ and $0_0 \otimes 0_1$ (and their conjugate) we reconstruct the multiplication table of $Oc(\mathcal{E}_9)$. As for the graph matrices of $\mathcal{E}_9$, the calculation is not straightforward, but imposing non-negative integer coefficients leads to a unique solution. The result is encoded in the 72 quantum symmetry matrices O_x of dimension 72×72.

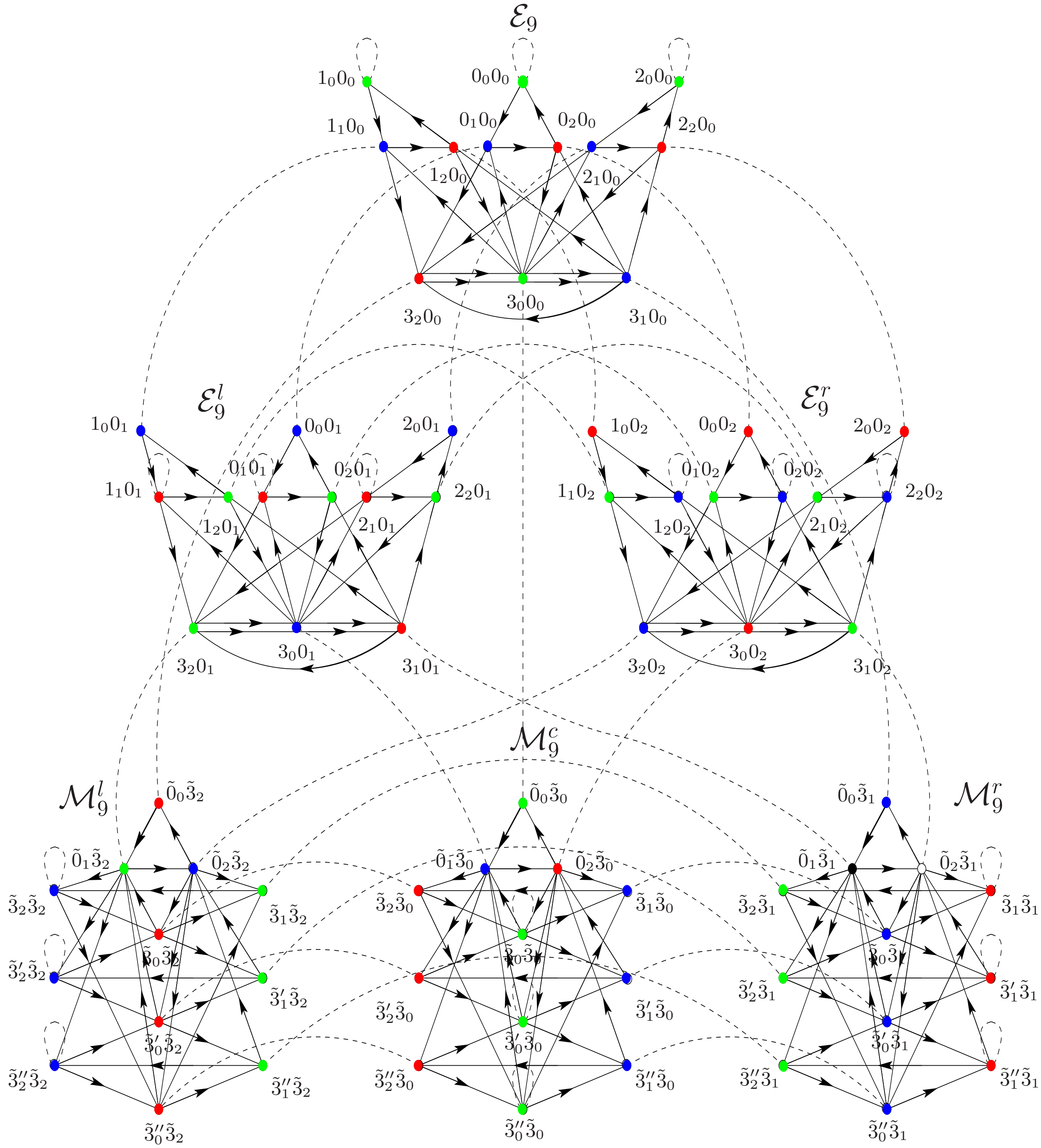

Figure 8: The Ocneanu graph $Oc(\mathcal{E}_9) = Oc(\mathcal{M}_9)$. The two left chiral generators are $0_1 \otimes 0_0$ and $0_2 \otimes 0_0$, the two right chiral generators are $0_0 \otimes 0_1$ and $0_0 \otimes 0_2$. The tensor product $a \otimes b$ is denoted with the shorthand notation ab.

Realization of the quantum symmetry algebra In order to have a compact (readable) description of these matrices and the multiplicative structures of the algebra of quantum symmetries, we propose the following realization of the algebra of quantum symmetries:

$$Oc \;=\; \text{“}\mathcal{E}_9 \otimes_{\mathbb{Z}_3} \mathcal{E}_9\text{”} \;\doteq\; (\mathcal{E}_9 \otimes_\rho \mathcal{E}_9) \oplus (\mathcal{M}_9 \otimes_\rho \mathcal{M}_9)\;, \tag{52}$$

where the notation $\otimes_\rho$ means that the tensor product is quotiented using the $\mathbb{Z}_3$ symmetry of graphs $\mathcal{E}_9$ and $\mathcal{M}_9$ in the following way. A basis of the quantum symmetry algebra is given by elements $\{a \otimes 0_i\,,\, \tilde{a} \otimes \tilde{3}_i\}$ for $i = 0,1,2$. The other elements of $\mathcal{E}_9 \otimes \mathcal{E}_9$ and $\mathcal{M}_9 \otimes \mathcal{M}_9$ are identified with basis elements $\{a \otimes 0_i\,,\, \tilde{a} \otimes \tilde{3}_i\}$ using the $\mathbb{Z}_3$ symmetry operators ρ_3 and ρ'_3 of graphs $\mathcal{E}_9$ and $\mathcal{M}_9$ and the induction-restruction rules between the two graph algebras, as follows:

- $a \otimes 1_i \;=\; a \otimes 1_0 \cdot 0_i \;=\; 1_0 \cdot a \otimes 0_i \;=\; \rho_3(a) \otimes 0_i$ (53)
- $a \otimes 2_i \;=\; a \otimes 2_0 \cdot 0_i \;=\; 2_0 \cdot a \otimes 0_i \;=\; (\rho_3)^2(a) \otimes 0_i$ (54)
- $a \otimes 3_i \;=\; \sum_{\tilde{a}} (E_{\tilde{0}_0})_{a\tilde{a}}\, \tilde{a} \otimes \tilde{3}_i$ (55)
- $\tilde{a} \otimes \tilde{3}'_i \;=\; \tilde{a} \otimes 1_0 \cdot \tilde{3}_i = 1_0 \cdot \tilde{a} \otimes \tilde{3}_i = \rho'_3(\tilde{a}) \otimes \tilde{3}_i$ (56)
- $\tilde{a} \otimes \tilde{3}''_i \;=\; \tilde{a} \otimes 2_0 \cdot \tilde{3}_i = 2_0 \cdot \tilde{a} \otimes \tilde{3}_i = (\rho'_3)^2(\tilde{a}) \otimes \tilde{3}_i$ (57)
- $\tilde{a} \otimes \tilde{0}_i \;=\; \sum_{a} (E^{tr}_{\tilde{0}_0})_{\tilde{a},a}\; a \otimes 0_i$ (58)

Here the matrix $E_{\tilde{0}_0}$ encodes the branching rules $\mathcal{E}_9 \hookrightarrow \mathcal{M}_9$ (obtained from matrices P^ℓ implementing the $\mathcal{E}_9$ (left) action on $\mathcal{M}_9$ as follows: $(E_{\tilde{b}})_{a\tilde{c}} = (P^\ell_a)_{\tilde{b}\tilde{c}}$). Explicitely, we have:

$$E_{\tilde{0}_0} = \begin{pmatrix}
1 & . & . & . & . & . & . & . & . & . & . & . \\
1 & . & . & . & . & . & . & . & . & . & . & . \\
1 & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & 1 & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & 1 & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & 1 & 1
\end{pmatrix}
\qquad
\begin{array}{lcl}
0_0 & \hookrightarrow & \tilde{0}_0 \\
1_0 & \hookrightarrow & \tilde{0}_0 \\
2_0 & \hookrightarrow & \tilde{0}_0 \\
3_0 & \hookrightarrow & \tilde{3}_0 + \tilde{3}'_0 + \tilde{3}''_0 \\
0_1 & \hookrightarrow & \tilde{0}_1 \\
1_1 & \hookrightarrow & \tilde{0}_1 \\
2_1 & \hookrightarrow & \tilde{0}_1 \\
3_1 & \hookrightarrow & \tilde{3}_1 + \tilde{3}'_1 + \tilde{3}''_1 \\
0_2 & \hookrightarrow & \tilde{0}_2 \\
1_2 & \hookrightarrow & \tilde{0}_2 \\
2_2 & \hookrightarrow & \tilde{0}_2 \\
3_2 & \hookrightarrow & \tilde{3}_2 + \tilde{3}'_2 + \tilde{3}''_2
\end{array} \tag{59}$$

The multiplication of the basis generators $\{a \otimes 0_i\,,\, \tilde{a} \otimes \tilde{3}_i\}$ is then naturally defined using the multiplication rules (50) and the projections (53–58). We introduce the matrices R^r defined from the right action of $\mathcal{E}_9$ on $\mathcal{M}_9$: $\tilde{b}\, a = \sum_{\tilde{c}} (P^r_a)_{\tilde{b}\tilde{c}}\, \tilde{c} = \sum_{\tilde{c}} (R^r_{\tilde{b}})_{a\tilde{c}}\, \tilde{c}$. It can be seen that the algebra $Oc(\mathcal{E}_9)$ is non commutative and isomorphic with the direct sum of 9 copies of 2×2 matrices and 36 copies of the complex numbers. With our parametrisation, the quantum symmetry matrices read:

$$
\begin{array}{lcl}
O_{a\otimes 0_0} & = & \begin{pmatrix} G_a & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & G_a & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & G_a & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & P^{\ell}_a & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & P^{\ell}_a & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & P^{\ell}_a \end{pmatrix} \\
O_{a\otimes 0_1} & = & \begin{pmatrix} \cdot & G_a & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & G_a & \cdot & \cdot & G_a\, E_0 \\ G_a & \cdot & \cdot & G_a\, E_0 & \cdot & \cdot \\ \cdot & P^{\ell}_a\, E_0^{tr} & \cdot & \cdot & P^{\ell}_a\,(\mathbb{1}+P^{\ell}_{1_0}) & \cdot \\ \cdot & \cdot & P^{\ell}_a\, E_0^{tr} & \cdot & \cdot & P^{\ell}_a \\ \cdot & \cdot & \cdot & P^{\ell}_a\,(\mathbb{1}+P^{\ell}_{2_0}) & \cdot & \cdot \end{pmatrix} \\
O_{a\otimes 0_2} & = & \begin{pmatrix} \cdot & \cdot & G_a & \cdot & \cdot & \cdot \\ G_a & \cdot & \cdot & G_a\, E_0 & \cdot & \cdot \\ \cdot & G_a & \cdot & \cdot & G_a\, E_0 & \cdot \\ \cdot & \cdot & P^{\ell}_a\, E_0^{tr} & \cdot & \cdot & P^{\ell}_a\,(\mathbb{1}+P^{\ell}_{1_0}) \\ \cdot & \cdot & \cdot & P^{\ell}_a\,(\mathbb{1}+P^{\ell}_{2_0}) & \cdot & \cdot \\ \cdot & P^{\ell}_a\, E_0^{tr} & \cdot & \cdot & P^{\ell}_a & \cdot \end{pmatrix} \\
O_{\tilde{a}\otimes\tilde{3}_0} & = & \begin{pmatrix} \cdot & \cdot & \cdot & R^r_{\tilde{a}} & \cdot & \cdot \\ \cdot & R^r_{\tilde{a}}\, E_0^{tr} & \cdot & \cdot & R^r_{\tilde{a}}(\mathbb{I}+P^{\ell}_{1_0}) & \cdot \\ \cdot & \cdot & R^r_{\tilde{a}} E_0^{tr} & \cdot & \cdot & R^r_{\tilde{a}}(\mathbb{1}+P^{\ell}_{1_0}) \\ H_{\overline{a}} & \cdot & \cdot & 2(H_{\tilde{a}} E_0) & \cdot & \cdot \\ \cdot & H_{\tilde{a}}(\mathbb{1}+G_{1_0}) & \cdot & \cdot & H_{\tilde{a}} E_0 & \cdot \\ \cdot & \cdot & H_{\tilde{a}}(\mathbb{1}+G_{1_0}) & \cdot & \cdot & H_{\tilde{a}} E_0 \end{pmatrix} \\
O_{\tilde{a}\otimes\tilde{3}_1} & = & \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & R^r_{\tilde{a}} & \cdot \\ \cdot & \cdot & R^r_{\tilde{a}}\, E_0^{tr} & \cdot & \cdot & R^r_{\tilde{a}} \\ \cdot & \cdot & \cdot & R^r_{\tilde{a}}(\mathbb{1}+P^{\ell}_{2_0}) & \cdot & \cdot \\ \cdot & H_{\tilde{a}}(\mathbb{1}+G_{2_0}) & \cdot & \cdot & H_{\tilde{a}}\, E_0 & \cdot \\ \cdot & \cdot & H_{\tilde{a}} & \cdot & \cdot & H_{\tilde{a}}\, E_0 \\ H_{\tilde{a}} & \cdot & \cdot & H_{\tilde{a}}\, E_0 & \cdot & \cdot \end{pmatrix} \\
O_{\tilde{a}\otimes\tilde{3}_2} & = & \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & R^r_{\tilde{a}} \\ \cdot & \cdot & \cdot & R^r_{\tilde{a}}(\mathbb{1}+P^{\ell}_{2_0}) & \cdot & \cdot \\ \cdot & R^r_{\tilde{a}}\, E_0^{tr} & \cdot & \cdot & R^r_{\tilde{a}} & \cdot \\ \cdot & \cdot & H_{\tilde{a}}(\mathbb{1}+G_{2_0}) & \cdot & \cdot & H_{\tilde{a}}\, E_0 \\ H_{\tilde{a}} & \cdot & \cdot & H_{\tilde{a}}\, E_0 & \cdot & \cdot \\ \cdot & H_{\tilde{a}} & \cdot & \cdot & H_{\tilde{a}}\, E_0 & \cdot \end{pmatrix}
\end{array}
\qquad (60)
$$

Triality t is well defined on this algebra: $t(a_i \otimes 0_j) = t(\tilde{a}_i \otimes \tilde{3}_j) = i + j\,(\mathrm{mod}3)$. The left chiral subalgebra (by definition the algebra generated by the left chiral generator $0_1 \otimes 0_0$) is $L = \{a \otimes 0_0\}$. The right chiral subalgebra (generated by $0_0 \otimes 0_1$) is $R = \{0_0 \otimes a\}$. With the projections (53-58), R correspondonds to the set of elements $\{0_0 \otimes 0_0, 1_0 \otimes 0_0, 2_0 \otimes 0_0, \tilde{0}_0 \otimes \tilde{3}_0, 0_0 \otimes 0_1, 1_0 \otimes 0_1, 2_0 \otimes 0_1, \tilde{0}_0 \otimes \tilde{3}_1, 0_0 \otimes 0_2, 1_0 \otimes 0_2, 2_0 \otimes 0_2, \tilde{0}_0 \otimes \tilde{3}_2\}$. The ambichiral subalgebra (by definition the intersection of L and R) is $A = \{0_0 \otimes 0_0, 1_0 \otimes 0_0, 2_0 \otimes 0_0\}$. The chiral operation C on the basis elements is defined by $C(u \otimes v) = (v \otimes u)$, for $u, v \in H = \mathcal{E}_9 \oplus \mathcal{M}_9$ (and using the projections (53-58)). The self-dual elements obey $C(u) = u$, they are the ones in figure 8 which are connected to themselves by the dashed line. A-elements are, in particular, self-dual.

One modular invariant and two graphs Starting from the modular invariant (41), we obtain the set of toric matrices W_{x0}, double fusion matrices $V_{\lambda\mu}$ and quantum symmetry matrices O_x, together with the corresponding Ocneanu graph. By an analysis of the later, it clearly appears that there are two graphs that are modules under the quantum symmetry algebra, the $\mathcal{E}_9$ and $\mathcal{M}_9$ graphs. Using the realization of the quantum symmetry algebra described above, the action is defined by:

$$Oc \times \mathcal{E}_9 \rightarrow \mathcal{E}_9 \qquad \left\{ \begin{array}{rcl} (a \otimes 0_i) \,.\, b &=& a \,.\, 0_i \,.\, b \\ (\tilde{a} \otimes \tilde{3}_i) \,.\, b &=& \tilde{a} \,.\, \tilde{3}_i \,.\, b \end{array} \right. \qquad \left\{ \begin{array}{rcl} S^{\mathcal{E}}_{x=a\otimes 0_i} &=& G_{0_i}\, G_a \\ S^{\mathcal{E}}_{x=\tilde{a}\otimes\tilde{3}_i} &=& R_{\tilde{3}_i}\, H_{\tilde{a}} \end{array} \right. \tag{61}$$

$$Oc \times \mathcal{M}_9 \rightarrow \mathcal{M}_9 \qquad \left\{ \begin{array}{rcl} (a \otimes 0_i) \,.\, \tilde{b} &=& a \,.\, 0_i \,.\, \tilde{b} \\ (\tilde{a} \otimes \tilde{3}_i) \,.\, \tilde{b} &=& \tilde{a} \,.\, \tilde{3}_i \,.\, \tilde{b} \end{array} \right. \qquad \left\{ \begin{array}{rcl} S^{\mathcal{M}}_{x=a\otimes 0_i} &=& P^{\ell}_{0_i}\, P^{\ell}_a \\ S^{\mathcal{M}}_{x=\tilde{a}\otimes\tilde{3}_i} &=& R_{\tilde{3}_i}\, H_{\tilde{a}} \end{array} \right. \tag{62}$$

We have therefore two quantum groupoïds associated with the initial modular invariant, constructed from the graphs $\mathcal{E}_9$ and $\mathcal{M}_9$. Setting $d^{\mathcal{E}}_\lambda = \sum_{a,b} (F^{\mathcal{E}}_\lambda)_{ab}$, $d^{\mathcal{E}}_x = \sum_{a,b} (S^{\mathcal{E}}_x)_{ab}$, $d^{\mathcal{M}}_\lambda = \sum_{a,b} (F^{\mathcal{M}}_\lambda)_{ab}$, $d^{\mathcal{M}}_x = \sum_{a,b} (S^{\mathcal{M}}_x)_{ab}$, we check the dimensional rules:

$$\dim(\mathcal{B}(\mathcal{E}_9)) = \sum_\lambda (d^{\mathcal{E}}_\lambda)^2 = \sum_x (d^{\mathcal{E}}_x)^2 = 518\,976 \,. \tag{63}$$

$$\dim(\mathcal{B}(\mathcal{M}_9)) = \sum_\lambda (d^{\mathcal{M}}_\lambda)^2 = \sum_x (d^{\mathcal{M}}_x)^2 = 754\,272 \,. \tag{64}$$

The rejected diagram In the first list of $SU(3)$-type graphs presented by Di Francesco and Zuber in [11], there were three graphs associated with the modular invariant (41): $\mathcal{E}_9$, $\mathcal{M}_9$ and another one, displayed on figure 9. This graph was later rejected by Ocneanu in [20], because some required cohomological property (written in terms of values for triangular cells) was not fullfilled. In other words, this graph gives rise to a module over the ring of $\mathcal{A}_9$, with the right properties, but the underlying category does not exist. Using our methods, this graph does not appear either. The reason is that it should be a module over the quantum symmetry algebra, but this is not the case: we can try to define such an action on this graph, but there is no solution when we impose the coefficients to be non-negative integers. In other words, imposing that a graph should define both a module over $\mathcal{A}$ and a module over Oc, with the expected non-negativity properties, amounts to impose, at least for the members of the $su(3)$ family, that both the module condition over $\mathcal{A}$ (only), together with the self - cell condition, should be satisfied. The methods described here provide therefore also powerful checks for graphs that are candidates to be described by quantum groupoïds associated with modular invariants.

Final Comment The Ocneanu graphs displayed in this paper ($Oc(E_6)$, $Oc(\mathcal{E}_5)$, $Oc(\mathcal{E}_9)$) have been first obtained by Ocneanu himself. For instance those associated with members of the $su(3)$ family were displayed on posters during the Bariloche conference (2000) but the full list never appeared in print. Several techniques [6, 25] allow one to recover some of them from the knowledge of the Di Francesco - Zuber diagrams. The present paper actually emerged from our wish to obtain the Ocneanu graphs $Oc(G)$ (and the graphs G themselves, of course) from the only data provided by the modular invariant.

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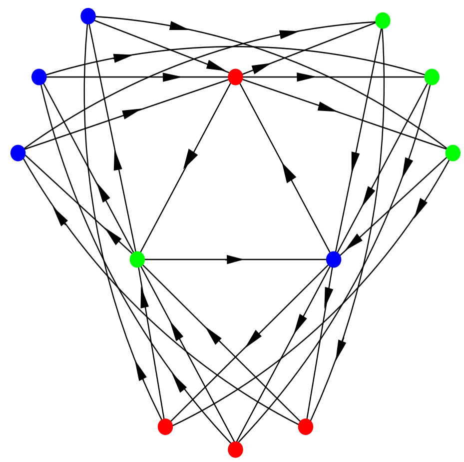

Figure 9: The rejected Di Francesco-Zuber graph.

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