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FERMIONIC DETERMINANT IN TWO AND FOUR DIMENSIONS

by

J.A. Mignaco and M.A. Rego Monteiro[§*]

Centro Brasileiro de Pesquisas Físicas - CNPq/CBPF
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

[§]Università di Torino
Istituto di Fisica Teorica
Corso Massimo d'Azeglio, 46
10125 - Torino - Italia

[*]Fellow of the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) - Brasil

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ABSTRACT

We discuss the fermionic determinant of the two-dimensional Schwinger model and QCD and a four-dimensional model with a pseudo-vectorial coupling. We observe that in both cases the Dirac operator can be expressed as a path-ordered product of the gauge field and compute exactly the fermionic determinant without reference to a particular gauge. We obtain the two point Green's function in all cases as a free particle two point function times a model dependent term.

Key-words: Effective lagrangian; Fermionic determinant; Axial anomaly; Wess-Zumino actions.

1 INTRODUCTION

The interest in the direct evaluation of fermionic determinants has been renewed after the advances originated in recent works by d'Adda, Davis and di Vecchia, Polyakov and Wiegman[1], Alvarez[2], Roskies[3], Gamboa Saraví, Muschietti, Schaposnik and Solomin[4]; Reuter[5], Naón[6] and others[7]. The evaluation of the fermionic determinant is an important step towards a complete solution of the fermionic field theory under consideration and important physical properties become exposed after integration over the matter fields. It also sheds light on formal properties derived from canonical procedures[3,6].

In this work we analize massless two-dimensional Schwinger model and QCD and a four dimensional Abelian and non-Abelian Dirac like theory with massless fermions interacting with an external field through a pseudo-vectorial coupling. In both cases, the Dirac operator can be expressed conveniently: by exponentials of an integral of the gauge field in the Abelian case, and by a path-ordered product of the gauge field in the non-Abelian case; we compute the fermionic determinant by the Alvarez method[2].

In two dimensional QCD the result of the determinant is a non-Abelian extension of the Schwinger mechanism[8] and an effective action which has one term with the form of the Wess-Zumino Lagrangian[9] written in terms of variables defined on curves as path-ordered products of the gauge field. In the four dimensional pseudo-vectorial model we also find this kind of Wess-Zumino Lagrangian plus several other terms.

We observe, in addition, that in all the cases analysed, similarly to the case discussed in Refs.[4,10], the calculated determinant is the Jacobian of a chiral transformation[11] and by this observation we are able to factorize the two-point correlation function in a Green's function of a massless free fermion times a factor which depends only on the external boson fields, exactly as happens in the soluble two-dimensional Abelian models such as the Schwinger[8] and Thirring[12] models.

In Section 2 we discuss the two-dimensional cases Abelian and non-Abelian, i.e., Schwinger model and two-dimensional QCD and in Section 3 the four dimensional pseudo-vectorial models, Abelian and non-Abelian.

2 TWO DIMENSIONAL CASE

2.1 Schwinger model[8]

The dynamics of the Euclidean model is determined by the Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ + A\!\!\!/)\psi \tag{1}$$

with $\gamma_\mu^\dagger = \gamma_\mu \quad , \quad A_\mu^\dagger = A_\mu$ and

$$i\gamma_\mu\gamma_5 = \varepsilon_{\mu\nu}\gamma_\nu \tag{2}$$

$$\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$$

Note that due to property (2) we may express the Dirac operator of this theory as:

$$D\!\!\!/ = i\partial\!\!\!/ + A\!\!\!/ = e^{-\gamma_5\varepsilon_{\alpha\beta}\int_{-\infty}^{x} A_\alpha dz_\beta}\; i\partial\!\!\!/\; e^{-\gamma_5\varepsilon_{\alpha\beta}\int_{-\infty}^{x} A_\alpha dz_\beta} \tag{3}$$

where the righ hand side of identity (3) is written in operator notation, i.e., let f be a function, then:

$$\partial_\mu f = (\partial_\mu f) + f\partial_\mu \tag{4}$$

We parametrize now the operator in (3) by using a real variable $t(0 \leq t \leq 1)$ and construct the operator $D\!\!\!/_t$ as:

$$D\!\!\!/_t = e^{-t\gamma_5\varepsilon_{\alpha\beta}\int_{-\infty}^{x} A_\alpha dz_\beta}\; i\partial\!\!\!/\; e^{-t\gamma_5\varepsilon_{\alpha\beta}\int_{-\infty}^{x} A_\alpha dz_\beta} \tag{5}$$

Note that $D\!\!\!/ = D\!\!\!/_{t=1}$ and $D\!\!\!/_{t=0} = i\partial\!\!\!/$.

This operator has the useful property that[2]:

$$\dot{D\!\!\!/}_t = \frac{d}{dt} D\!\!\!/_t = fD\!\!\!/_t + D\!\!\!/_t f \tag{6}$$

with:

$$f = -\gamma_5\varepsilon_{\alpha\beta}\int_{-\infty}^{x} A_\alpha dz_\beta \tag{7}$$

We regularize the determinant of $D\!\!\!/_t$ by the proper-time method[13]:

$$\ell n \det D\!\!\!/_t^2 = -\int_\varepsilon^\infty \frac{ds}{s} \mathrm{Tr}[\exp(-sD\!\!\!/_t^2)] \tag{8}$$

with ε an ultraviolet cutoff in the proper-time method.

Differentiating (8) with respect to t, using property (6) and the cyclic property of the functional trace we get:

$$\frac{d}{dt}\,\ell n\,\det \not{D}_t^2 = 2\int_\varepsilon^\infty ds\ \mathrm{Tr}[\not{D}_t\dot{\not{D}}_t \exp(-s\not{D}_t^2)] = 4\int_\varepsilon^\infty ds\ \mathrm{Tr}[f\not{D}_t^2\ \exp(-s\not{D}_t^2)] =$$

$$= -4\int_\varepsilon^\infty ds\ \frac{d}{ds}\ \mathrm{Tr}[f\exp(-s\not{D}_t^2)] = 4\,\mathrm{Tr}[f\exp(-\varepsilon\not{D}_t^2)] \qquad (9)$$

Integrating (9) with respect to t we get:

$$\ell n\,\frac{\det \not{D}}{\det i\not{\partial}} = 2\int_0^1 dt\int d^2x\ \mathrm{tr}_\gamma[f < x|\exp(-\varepsilon\not{D}_t^2)|x>] \qquad (10)$$

with tr_γ denoting the trace over the Dirac γ matrices.

The diagonal part of the heat kernel which we have in the integrand of eq. (10) for the operator in consideration has the asymptotic small ε expansion(14) tabulated(15). The square operator D_t^2 can be written as:

$$D_t^2 = (i\partial_\mu + tA_\mu)^2 - t\varepsilon_{\mu\nu}\gamma_5 A_\nu \qquad (11)$$

and the diagonal part of the heat kernel for this operator has the asymptotic small ε expansion given as(15):

$$<x|\exp(-\varepsilon\not{D}_t^2)|x> \underset{\varepsilon\to 0}{\sim} \frac{1}{4\pi\varepsilon} + \frac{t}{4\pi}\ \varepsilon_{\mu\nu}\gamma_5\partial_\mu A_\nu + O(\varepsilon) \qquad (12)$$

Substituting this result together with (7) in (10) and neglecting surface terms we get:

$$\ell n\,\frac{\det \not{D}}{\det i\not{\partial}} = -\frac{1}{2\pi}\int d^2x\ A_\mu A_\mu \qquad (13)$$

It is worthwhile to mention that the correlation functions for this theory can be easily computed with this method by observing that the determinant is the Jacobian of a particular chiral transformation involving the gauge field, as is done in reference (4).

2.2 Two dimensional QCD

The lagrangian of the Euclidean two-dimensional QCD is given by:

$$\mathcal{L}_{QCD} = -\frac{1}{4}\, F^a_{\mu\nu}F^a_{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ + A\!\!\!/)\psi \tag{14}$$

In order to proceed analogously to what we have done in the Abelian case, due to the non-Abelian nature we must consider now path ordered products and not the exponential of an integral. We note that the Dirac operator in this case can be written as

$$D\!\!\!/ = i\partial\!\!\!/ + A\!\!\!/ = \lim_{\Delta^i \to 0} U(c^x;x,-\infty)\, i\partial\!\!\!/ U(c^x;-\infty,x) \tag{15}$$

with $\lim\limits_{\Delta^i \to 0}$ denoting the limit for continuous curve and

$$\begin{aligned} U(c^x;x,-\infty) &= e^{-\gamma_5\varepsilon_{\alpha\beta}A^x_\alpha\Delta^x_\beta}\, e^{-\gamma_5\varepsilon_{\alpha\beta}A^{x-1}_\alpha\Delta^{x-1}_\beta} \ldots e^{-\gamma_5\varepsilon_{\alpha\beta}A^{-\infty}_\alpha\Delta^{-\infty}_\beta} \\ U(c^x;-\infty,x) &= e^{-\gamma_5\varepsilon_{\alpha\beta}A^{-\infty}_\alpha\Delta^{-\infty}_\beta} \ldots e^{-\gamma_5\varepsilon_{\alpha\beta}A^{x-1}_\alpha\Delta^{x-1}_\beta}\, e^{-\gamma_5\varepsilon_{\alpha\beta}A^x_\alpha\Delta^x_\beta} \end{aligned} \tag{16}$$

Here Δ^i_β is the i-th partition of the curve c^x which goes from $-\infty$ to x and A^x_α is the field A_α in the space-time point x.

Now, in order to construct an operator depending on a real variable t $(0 \leq t \leq 1)$ analogously to what we did in the Abelian case we define:

$$U_t(c^x;x_t,-\infty) = e^{-t\gamma_5\varepsilon_{\alpha\beta}A_\alpha^x\Delta_\beta^x}\, e^{-\gamma_5\varepsilon_{\alpha\beta}A_\alpha^{x-1}\Delta_\beta^{x-1}}\ldots e^{-\gamma_5\varepsilon_{\alpha\beta}A_\alpha^{-\infty}\Delta_\beta^{-\infty}}$$
$$U_t(c^x;-\infty,x_t) = e^{-\gamma_5\varepsilon_{\alpha\beta}A_\alpha^{-\infty}\Delta_\beta^{-\infty}}\ldots e^{-\gamma_5\varepsilon_{\alpha\beta}A_\alpha^{x-1}\Delta_\beta^{x-1}}\, e^{-t\gamma_5\varepsilon_{\alpha\beta}A_\alpha^x\Delta_\beta^x} \tag{17}$$

Notice that the parameter t only enters with the partition Δ_β^x. We then define the operator $\not{D}_t$ depending on the parameter t as:

$$\not{D}_t = U_t(c^x;x_t,-\infty)\, i\not{\partial}\, U_t(c^x;-\infty,x_t) \tag{18}$$

such that:

$$\not{D} = i\not{\partial} + \not{A} = \lim_{\Delta^i\to 0} \not{D}_{t=1} \tag{19}$$

The operator $\not{D}_t$ has the property that[2]:

$$\dot{\not{D}}_t = \frac{d}{dt}\not{D}_t = f\not{D}_t + \not{D}_t f \tag{20}$$

with:

$$f = -\gamma_5\varepsilon_{\alpha\beta}A_\alpha^x\Delta_\beta^x \tag{21}$$

In order to write the relation (20) in a form that makes clear the limit for continuous curve, let us define:

$$V_t = V_t(c^x;-\infty,x_t) \tag{22}$$

with:

$$V_t(c^x;-\infty,x_t) = e^{-\varepsilon_{\alpha\beta}A_\alpha^{-\infty}\Delta_\beta^{-\infty}} \ldots e^{-\varepsilon_{\alpha\beta}A_\alpha^{x-1}\Delta_\beta^{x-1}} e^{-t\varepsilon_{\alpha\beta}A_\alpha^{x}\Delta_\beta^{x}} \quad . \tag{23}$$

Then f can be written as:

$$f = \gamma_5 V_t^{-1}\partial_t V_t \tag{24}$$

with $\partial_t = \partial/\partial t$.

Regularizing the determinant of $\not{D}_t$ by the proper-time method[15] we have:

$$\ell n \det \not{D}_t^2 = -\int_\varepsilon^\infty \frac{ds}{s}\, \mathrm{Tr}[\exp(-s\not{D}_t^2)] \tag{25}$$

Differentiating (25) with respect to t, using property (20), the cyclic property of the functional trace and integrating with respect to t from 0 to 1, analogously to what we did in the Abelian case (Eqs. (9) and (10)), we get

$$\ell n \frac{\det \not{D}_{t=1}}{\det i\not{\partial}} = 2\int_0^1 dt \int d^2x \ \mathrm{tr}_{c\times\gamma}\ [f\langle x|\exp(-\varepsilon\not{D}_t^2)|x\rangle] \tag{26}$$

with $\mathrm{tr}_{c\times\gamma}$ the trace over colour and Dirac γ matrices indices.

Let us define now:

$$\not{D}_t = i\,\gamma_\mu(\partial_\mu + B_\mu^t) = i\,\gamma_\mu D_\mu^t \tag{27}$$

with

$$\not{B}^t = U_t(c^x;x_t,-\infty)[\not{\partial} U_t(c^x;-\infty,x_t)] \quad . \tag{28}$$

In this case, the diagonal part of the heat kernel for the operator $\not{D}_t$ has the asymptotic small ε expansion[14] given as[15]:

$$\langle x|\exp(-\varepsilon\not{D}_t^2)|x\rangle \underset{\varepsilon\to 0}{\sim} \frac{1}{4\pi}\left[\frac{1}{\varepsilon}+\not{\partial}\not{B}^t+\not{B}^t\not{B}^t-\partial_\mu B_\mu^t-B_\mu^t B_\mu^t+0(\varepsilon)\right] \quad (29)$$

Substituting (29) in (26) we obtain:

$$\ell n\,\frac{\det \not{D}_{t=1}}{\det i\not{\partial}} = \frac{1}{2\pi}\int d^2x\int_0^1 dt\ \underset{cx\gamma}{\mathrm{tr}}\ [(\not{\partial}f)\not{B}^t + f\,\not{B}^t\not{B}^t] \quad (30)$$

but using (28) we have:

$$\frac{1}{2}\frac{d}{dt}\underset{cx\gamma}{\mathrm{tr}}(\not{B}^t\not{B}^t) - \underset{cx\gamma}{\mathrm{tr}}(f\,\not{B}^t\not{B}^t) = \underset{cx\gamma}{\mathrm{tr}}(f\,\not{B}^t\not{B}^t) + \underset{cx\gamma}{\mathrm{tr}}((\not{\partial}f)\not{B}^t) \quad (31)$$

and with (30) and (31) we may partially integrate over t obtaining:

$$\ell n\,\frac{\det \not{D}_{t=1}}{\det i\not{\partial}} = \frac{1}{2\pi}\int d^2x\{\frac{1}{2}\underset{cx\gamma}{\mathrm{tr}}(\not{B}\not{B}) - \int_0^1 dt\ \underset{cx\gamma}{\mathrm{tr}}(\not{B}^t\not{B}^t f)\} \quad (32)$$

with $\not{B}=\not{B}^{t=1}$.

We note at this point that if the group is Abelian we would obtain the result of two-dimensional QED since the last term in the right hand side of (32) has null trace and the first term is the Schwinger mechanism[6,8].

We may still calculate the traces over the Dirac γ matrices. For this purpose let us recall the definition of the right and left handed fermion fields:

$$\psi_R = \frac{(1+\gamma_5)}{2}\psi \qquad \bar{\psi}_R = \bar{\psi}\frac{(1-\gamma_5)}{2}$$
$$\psi_L = \frac{(1-\gamma_5)}{2}\psi \qquad \bar{\psi}_L = \bar{\psi}\frac{(1+\gamma_5)}{2} \tag{33}$$

In terms of these fields we may write:

$$\bar{\psi}\not{D}_t\psi = \bar{\psi}_R U_t(c^x;x_t,-\infty)\, i\not{\partial} U_t(c^x;-\infty,x_t)\psi_R + (R \to L) \tag{34}$$

Using the chirality of left and right handed fields:

$$\bar{\psi}\not{D}_t\psi = \psi_R V_t^{-1}\, i\not{\partial}\, V_t\psi_R + \bar{\psi}_L V_t^{\dagger}\, i\not{\partial}\,(V_t^{\dagger})^{-1}\,\psi_L \tag{35}$$

with V_t defined by Eq. (22). Now, from (33) and (35) we obtain:

$$\bar{\psi}\not{D}_t\psi = \bar{\psi}(i\not{\partial}+\not{V}_t+\not{A}_t)\psi \tag{36}$$

where:

$$\not{V}_t = \frac{1}{2}\,\{V_t^{-1} i[\not{\partial}V_t] + V_t^{\dagger} i[\not{\partial}(V_t^{\dagger})^{-1}]\}$$
$$\not{A}_t = \frac{\gamma_5}{2}\,\{V_t^{\dagger} i[\not{\partial}(V_t^{\dagger})^{-1}] - V_t^{-1} i[\not{\partial}V_t]\} \tag{37}$$

It is easy to show that in the limit for continuous curve the term $\not{V}_t$ goes to zero, remaining only $\not{A}_t$ which gives the correct result for $\lim\limits_{\Delta\to 0} \not{D}_{t=1} = i\not{\partial}+\not{A}$. Then substituting (36) and (37) in (32), performing the trace over the Dirac γ matrices and the limit for continuous curve, we obtain:

$$-S_{eff} = \ln \frac{\det \not{D}}{\det i\not{\partial}} = -\frac{1}{8\pi}\int d^2x \; \underset{c}{tr} \; \{(V^{\dagger}[\partial_{\mu}(V^{\dagger})^{-1}] - V^{-1}[\partial_{\mu}V])^2\} \; +$$

$$+ \frac{i}{4\pi}\varepsilon_{\mu\nu}\int d^2x \int_0^1 dt \; \underset{c}{tr}\{(V_t^{\dagger}[\partial_{\mu}(V_t^{\dagger})^{-1}] - V_t^{-1}[\partial_{\mu}V_t])(V_t^{\dagger}[\partial_{\nu}(V_t^{\dagger})^{-1}] - V_t^{-1}[\partial_{\nu}V_t])V_t^{-1}\partial_t V_t\} \tag{38}$$

with $V=V_{t=1}$

In the second integrand of equation (38) we have the following term:

$$I_{w-z} = \frac{i}{4\pi}\varepsilon_{\mu\nu}\int d^2x \int_0^1 dt \; \underset{c}{tr}\{V_t^{-1}[\partial_{\mu}V_t]V_t^{-1}[\partial_{\nu}V_t]V_t^{-1}[\partial_t V_t]\} \tag{39}$$

Since we are considering the space-time as a large sphere and $\pi_2(SU(N))=0$ $(N \geq 2)$, this last integral can be considered as being performed on a solid sphere (B) with unitary radius, having S^2 as frontier, with the the two space-time and the parameter t $(0 \leq t \leq 1)$ as coordinates; we then may write:

$$I_{w-z} = \frac{i}{2\pi}\varepsilon_{ijk}\int d^3x \; \underset{c}{tr}[V_t^{-1}(\partial_i V_t)V_t^{-1}(\partial_j V_t)V_t^{-1}(\partial_k V_t)] \tag{40}$$

taking into account that for $i=1,2,\partial_i$ represent the space-time de̲rivatives and for $i=3$, $\partial_i=\partial/\partial t$. It is important to observe that I_{w-z} has the form of the two-dimensional analogue of the Wess-Zumino functional[9] written in terms of variables defined on curves as path-ordered products of the gauge field.

In order to study the correlation functions of the theory we will perform the following transformation on the fermion field:

$$\psi(x) \to \lim_{\Delta\to 0} U^{-1}(c^x;x,-\infty)\psi(x) = \lim_{\Delta\to 0} U_+^{-1}\psi(x)$$
$$\bar\psi(x) \to \bar\psi(x) \lim_{\Delta\to 0} U^{-1}(c^x,-\infty,x) = \bar\psi \lim_{\Delta\to 0} U_-^{-1} \tag{41}$$

with U given in (16). We see by this transformation that the fermionic part of the lagrangian goes to a free field theory which gives, when integrated over the fermion degrees of freedom, a field independent constant times a Jacobian which turns to be the fermionic determinant that we have computed.

Let us consider now the generating functional with the fermion sources θ and $\bar\theta$:

$$Z(\bar\theta,\theta) = \int DA\, D\bar\psi\, D\psi\, \exp\{-\int d^2x(\mathcal{L}_{QCD} + \bar\theta\psi + \bar\psi\theta)\} \tag{42}$$

Perform the transformation (41):

$$Z(\bar\theta,\theta) = \int J\, DA\, D\bar\psi\, D\psi\, \exp\{-\int d^2x(\bar\psi i\not\partial\psi + \bar\theta \lim_{\Delta\to 0} U_+^{-1}\psi + \bar\psi \lim_{\Delta\to 0} U_-^{-1}\theta - $$
$$- \frac{1}{4} F^a_{\mu\nu}F^a_{\mu\nu})\} \tag{43}$$

with:

$$\ell n\, J = -S_{eff} \tag{44}$$

where S_{eff} is given by (38). Then differentiating with respect to the sources and turning them off, we obtain for the two-point correlation function:

$$\langle\psi(x)\bar\psi(y)\rangle = \langle\psi(x)\bar\psi(y)\rangle_0 \int DA \lim_{\Delta\to 0} U^{-1}(c^x;x,-\infty) \lim_{\Delta\to 0} U^{-1}(c^y;-\infty,y)\,\cdot$$
$$\cdot\ \exp\{-S_{eff} + \int d^2x\, \frac{1}{4} F^a_{\mu\nu}F^a_{\mu\nu}\} \tag{45}$$

with $<\psi(x)\bar{\psi}(y)>_0$ the free fermion Green's function. We find then a quantum decoupling for the two-point fermion correlation function exactly as occurs in two-dimensional QED[4].

3 FOUR DIMENSIONAL CASE

We are going now to consider a four-dimensional model, Abelian and non-Abelian, with fermions interacting with an external vector field by means of a pseudo-vectorial coupling. This model can be considered as a simplified version of the Weinberg - Salam model for massless fermions.

3.1 Abelian Case

The Lagrangian of the model is given by:

$$\mathcal{L}_M = \bar{\psi} i(\not{\partial} - \gamma_5 \not{A})\psi = \bar{\psi}\not{D}\psi \tag{46}$$

The Dirac operator $\not{D}$ can be written as:

$$\not{D} = i(\not{\partial} - \gamma_5 \not{A}) = e^{\gamma_5 \int_{-\infty}^{x} A.dz} \, i\not{\partial} \, e^{\gamma_5 \int_{-\infty}^{x} A.dz} \tag{47}$$

in operator notation, see Eq. (4).

The effective action of the theory is defined as:

$$e^{-S_{eff}(A)} = \int D\bar{\psi} D\psi \, \exp(-\int d^4x \, \mathcal{L}_M) = e^{\ln \det \not{D}} \tag{48}$$

In order to compute this effective action we parametrize the operator $\not{D}$ by a parameter t $(0 \leq t \leq 1)$ as:

$$\not{D}_t = e^{t\gamma_5 \int_{-\infty}^{x} A.dz} \, i\not{\partial} \, e^{t\gamma_5 \int_{-\infty}^{x} A.dz} \tag{49}$$

This operator has the useful property[2]:

$$\dot{\not{D}}_t = \frac{d}{dt} \not{D}_t = \gamma_5 \beta(x) \not{D}_t + \not{D}_t \beta(x) \gamma_5 \tag{50}$$

with

$$\beta(x) = \int_{-\infty}^{x} A.dz \tag{51}$$

We regularize the fermionic determinant by the proper-time method[13]:

$$\ell n \det \not{D}_t^2 = -\int_{\varepsilon}^{\infty} \frac{ds}{s} \, \mathrm{Tr}[\exp(-s\not{D}_t^2)] \tag{52}$$

Now, analogously to what we have done in Section 2, i.e, differentiating (52) with respect to t, using (50), (51) and the cyclic property of the functional trace, we obtain:

$$\frac{d}{dt} \ell n \det \not{D}_t^2 = 4\int d^4x \, \underset{c}{\mathrm{tr}}[\gamma_5 \beta(x) <x|\exp(-\varepsilon\not{D}_t^2)|x>] \tag{53}$$

According to reference (16) we have:

$$\underset{\gamma}{\mathrm{tr}}[\gamma_5 \beta(x) <x|\exp(-\varepsilon\not{D}_t^2)|x>] = \frac{\beta(x)}{32\pi^2} \, (G_t^{(1)} + G_t^{(2)}) \tag{54}$$

with:

$$G_t^{(1)} = \frac{4}{3}\, t^2\, \varepsilon_{\mu\alpha\beta\gamma}\, \partial_\mu (A_\alpha \partial_\beta A_\gamma)$$
$$G_t^{(2)} = \frac{4}{3}\, \partial_\mu (4t^3 A_\mu A^2 + t\partial^2 A_\mu) \tag{55}$$

Substituting (54) and (55) in (53), neglecting surface terms and integrating over t from 0 to 1, we get:

$$-S_{eff} = \ell n\, \frac{\det \not{D}}{\det i\not{\partial}} = -\frac{1}{6\pi^2}\int d^4x\, (A^4 + \frac{1}{2}\, A_\mu \partial^2 A_\mu) \tag{56}$$

The correlation functions can be studied in a similar way as we have done in the end of Section 2. The generating functional with fermionic sources θ and $\bar{\theta}$ is given by:

$$Z(\theta,\bar{\theta}) = \int D\bar{\psi}D\psi\, \exp\{-\int d^4x(\mathcal{L}_M + \bar{\theta}\psi + \bar{\psi}\theta)\} \tag{57}$$

We perform the chiral transformation:

$$\psi \rightarrow e^{-\gamma_5 \beta(x)}\psi$$
$$\bar{\psi} \rightarrow \bar{\psi} e^{\gamma_5 \beta(x)} \tag{58}$$

Taking into account that the Jacobian coming from this chiral change of variables is equal to the fermionic determinant we have computed in this section, by the same reasons we have discussed in the last section, the generating functional after this chiral change goes to:

$$Z(\theta,\bar{\theta}) = \int D\bar{\psi}D\psi \; \exp\{-\int d^4x(\mathcal{L}^M_{eff}+\bar{\theta}e^{-\gamma_5\beta(x)}\psi + \bar{\psi}\,e^{-\gamma_5\beta(x)}\theta + \bar{\psi}i\partial\!\!\!/\psi)\} \tag{59}$$

where

$$\mathcal{L}^M_{eff} = \frac{1}{6\pi^2}\,(\frac{1}{2}\,A_\mu\partial^2A_\mu+A^4) \tag{60}$$

The two-point Green's function can now be easily computed from (59) and gives:

$$<\psi(x)\bar{\psi}(y)> = <\psi(x)\bar{\psi}(y)>_o\; e^{-\gamma_5\beta(x)}\,e^{-\gamma_5\beta(y)}\,e^{-S_{eff}} \tag{61}$$

with $<\psi(x)\bar{\psi}(y)>_o$ denoting the free fermion Green's function.

2.2 Non Abelian case

The Lagrangian of the model is given by (46) but in this case the fields which appear are defined on some non-Abelian group. Following what we have done for two-dimensional QCD we define a parametrized operator $\not{D}_t$ with t a real parameter $(0 \le t \le 1)$ as:

$$\not{D}_t = U_t(c^x;x_t,-\infty)\,i\partial\!\!\!/ U_t(c^x;-\infty,x_t) \tag{62}$$

written in operator notation (see Eq. (4)), where:

$$\begin{aligned} U_t(c^x;x_t,-\infty) &= e^{t\gamma_5A^x.\Delta^x}\,e^{\gamma_5A^{x-1}.\Delta^{x-1}}\ldots e^{\gamma_5A^{-\infty}.\Delta^{-\infty}} \\ U_t(c^x;-\infty,x_t) &= e^{\gamma_5A^{-\infty}.\Delta^{-\infty}}\ldots e^{\gamma_5A^{x-1}.\Delta^{x-1}}\,e^{t\gamma_5A^x.\Delta^x} \end{aligned} \tag{63}$$

with A^x_α and Δ^i_β defined in the same way we have used for two-dimensional QCD. Note that

$$\not{D} = i(\not{\partial}-\gamma_5 \not{A}) = \lim_{\Delta^i \to 0} \not{D}_{t=1} \tag{64}$$

Again, the operator $\not{D}_t$ has the useful property that[2]:

$$\dot{\not{D}}_t = \frac{d}{dt} \not{D}_t = \gamma_5 \beta(x) \not{D}_t + \not{D}_t \beta(x) \gamma_5 \tag{65}$$

with

$$\beta(x) = V_t^{-1}(\partial_t V_t) \tag{66}$$

where $\partial_t = \partial/\partial t$ and:

$$V_t = V_t(c^x;-\infty,x_t) = e^{A^{-\infty}.\Delta^{-\infty}} \ldots e^{A^{x-1}.\Delta^{x-1}} \, e^{tA^x.\Delta^x} \tag{67}$$

Now, analogously to what we did in Section 2, regularizing the determinant of $\not{D}_t$ by the proper-time method, differentiating with respect to t, using property (65), the cyclic property of the functional trace and finally integrating over t from zero to one, we obtain:

$$\ln \frac{\det \not{D}_{t=1}}{\det i\not{\partial}} = 2\int_0^1 dt \int d^4x \, \underset{c x \gamma}{\mathrm{tr}} \, [\gamma_5 \beta(x) \langle x | \exp(-\varepsilon \not{D}_t^2) | x \rangle] \tag{68}$$

By the use of left and right handed spinor fields we may write:

$$\not{D}_t = \not{L}_t + \not{R}_t \tag{69}$$

with:

$$\not{\alpha}_t = \frac{1}{2}\{V_t^{-1} i[\not{\partial} V_t] + V_t^{\dagger} i[\not{\partial}(V_t^{\dagger})^{-1})\}$$

$$\not{\beta}_t = -\frac{\gamma_5}{2}\{V_t^{-1} i[\not{\partial} V_t] - V_t^{\dagger} i[\not{\partial}(V_t^{\dagger})^{-1})\} \equiv -i\gamma_5\gamma_\mu G_\mu^t \quad (70)$$

It is easy to see that in the limit of a continuous, curve, i.e. $\lim\limits_{\Delta i \to 0}$, the term $\not{\alpha}_t$ goes to zero. Then according to reference(16) we have:

$$-S_{eff} = \ell n \frac{\det \not{D}}{\det i\not{\partial}} = \frac{1}{16\pi^2}\int_0^1 dt \int d^4x \, \mathop{tr}_c [\beta(x)(H^{(1)} + H^{(2)}] \quad (71)$$

with

$$H^{(1)} = 4\epsilon_{\mu\nu\rho\sigma}[\frac{1}{4} V_{\mu\nu}^t V_{\rho\sigma}^t + \frac{1}{12} A_{\mu\nu}^t A_{\rho\sigma}^t - \frac{2}{3}(G_\mu^t G_\nu^t V_{\rho\sigma}^t + G_\mu^t V_{\nu\rho}^t G_\sigma^t + V_{\mu\nu}^t G_\rho^t G_\sigma^t) + \frac{8}{3} G_\mu^t G_\nu^t G_\rho^t G_\sigma^t]$$

$$H^{(2)} = 2[\frac{4}{3}\{\partial_\mu G_\nu^t + \partial_\nu G_\mu^t, G_\mu^t G_\nu^t\} - \frac{2}{3}\{\partial_\mu G_\mu^t, (G_\mu^t)^2\} + \frac{4}{3}[G_\mu^t, \partial_\lambda V_{\mu\lambda}^t] - \frac{1}{3}[A_{\mu\lambda}^t, V_{\mu\lambda}^t] + \frac{2}{3}\partial^2(\partial_\mu G_\mu^t) + 4G_\lambda^t(\partial_\mu G_\mu^t)G_\lambda^t] \quad (72)$$

where

$$V_{\mu\nu}^t \equiv [G_\mu^t, G_\nu^t]$$

$$A_{\mu\nu}^t \equiv \partial_\mu G_\nu^t - \partial_\nu G_\mu^t \quad (73)$$

Note that we also have taken the limit for a continuous curve.

It is important to observe that from all the terms in $H^{(1)}$ we obtain a contribution in (71) with the same form of

Wess-Zumino functional[9] written in terms of V_t.

Again, as we have done to study the two-point Green's function in two-dimensional QCD and in the Abelian four-dimensional model we can easily show that the two point Green's function is:

$$<\psi(x)\bar{\psi}(y)> = <\psi(x)\bar{\psi}(y)>_o \lim_{\Delta^i\to 0} U^{-1}(c^x;x,-\infty) \lim_{\Delta^i\to 0} U^{-1}(c^x;-\infty,y)\, e^{-S_{eff}} \quad (74)$$

where

$$U^{-1}(c^x;x,-\infty) = U^{-1}_{t=1}(c^x;x_{t=1},-\infty) \quad (75)$$

and we also see in this case the decoupling of the fermionic sector.

4 CONCLUSION

We have presented in this work the results of calculations on several field theoretical models performed through a method which can be considered as a new extension of the one introduced by Schwinger more than thirty years ago. The ingredients are basically the same for the regularization, the main improvement being the adoption of developments from functional analysis[14,15].

The calculation of axial anomalies proceeds quite naturally with this procedure. At the same time, is an alternative that encompasses the procedure introduced by Fujikawa[9].

The method is also an alternative to the calculation through the regularization of operator determinants by ζ-functions[5]. It seems easier to handle, and in some cases physical information appears more transparently.

By this we specially mean that we have been able to introduce almost naturally string-like extended structures. The results seem to improve on previous work, and look in better agreement with what should be expected on theoretical grounds[17] as a natural path to develop an effective lagrangian for hadronic systems at low energies, from the starting point of a non-abelian gauge invariant lagrangian. The path dependence of these structures is easily disposed of by construction.

A finer point to be better understood is the connection with the results of this article with the "true" (yet unknown) mechanism operating for the breaking of chiral symmetry. The effective lagrangians referred above condense a lot of information obtained through current algebras. The content of the lagrangians coming from the fermion determinant is determined by the non invariance of the measure of the functional integral under chiral transformations, and is essentially determined by the small distance behaviour of the relevant operators. This curious interplay between short distance and long distance features seems to be a clue for the understanding of the dynamical content of gauge theories.

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