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ON THE PHYSICAL PROBLEM OF SPATIAL DIMENSIONS: AN ALTERNATIVE PROCEDURE TO STABILITY ARGUMENTS+*

by

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Abstract

Why is space three-dimensional? The first answer, entirely based on Physics, to this question, was given by Ehrenfest in 1971, who showed that the n-dimensional solar system can be stable only for n = 3. Following this approach, Tangherlini showed, in 1963, from quantum mechanics, that this is the case also for Hydrogen atoms. In the present work, we critically review the epistemological consequences of the use of the stability postulate to derive spatial dimensionality. An alternative procedure to get at it, in which the stability postulate (and the implicit singularities in three-dimensional physics) are not an essential part of the arguments, is proposed. We are then led to discuss the n-dimensional quantum theory as expressed in Plank's law, de Broglie relation and Heisenberg uncertainty relation. From these results we discuss a possible experiment, based on thermal neutron diffraction by crystals, to directly *measure* the dimensionality of space. Also, from the black-body theory, as applied to the Sun, it is shown that the existence of life on Earth is possible only if space, in the neighborhood of the Sun, has been three-dimensional in a cosmic time scale. Finally, the distinguished rôle of Maxwell's electromagnetic theory in the determination of space dimensionality is stressed. "Metric" versus "topological" arguments are compared and shown to apply respectively to "matter" and "fields".

Key-words: Dimension; Space dimensionality; Anthropic principle; Stability of the world; Neutron diffraction; Wein's law.

1. *Introduction.*

In this paper we discuss the dimensionality of space as a physical problem.

At first sight this problem could be approached, in a fruitful way by simply asking the question: why is space three-dimensional? However, on second thought, it is clear that this formulation is narrow minded since the three-dimensionality of space is assumed as something given *a priori* perhaps by our sense organs, especially vision. We shall come back to this point later, but now it suffices to say that our interaction with the external world (via our vision) is essentially electromagnetic and that Electromagnetism implies a three-dimensional world, as we will see below. Now, in Physics we are not restricted to our direct experience of the external world, i.e., to our sensory perception. So we can investigate the problem in a much more profound way by freeing ourselves from our sensory prejudices and trying to answer the following complementary questions: i) How does it become manifest in the fundamental laws of Physics that space *has* 3 dimensions? ii) How do the fundamental laws of Physics *entail* space dimensionality? These two questions will be discussed throughout this paper both by critically reviewing the existing literature and by proposing new approaches to this problem.

Some readers may find somewhat "unpleasant" in this paper the several digressions and some apparently "unnecessary" repetitions. We hope that this feeling will disappear by the time we come to the conclusions, with the realization that these digressions and footnotes are indeed necessary and that, often, they are there to clarify points which would, otherwise, remain

somewhat obscure. In this perspective, they are, in fact, fundamental to our final discussion.

Before proceeding and entering more deeply into our subject we must first clarify some points which more plainly define our conceptual framework.

We begin by considering that the dimensionality of space *is not a contingent feature*. To accept this means that one must search for a general methodology capable of determining it. A fundamental ingredient is necessarily the possibility of thinking about higher dimensional space, which is provided by the works of Lobačevskij, Bolyai, Gauss, Cayley, Grassmann, and Riemann [1], but as we will see below this is not sufficient. Although at early times the physical soundness of this kind of generalizations was continuously questioned[1], there is nowadays a kind of general consensus that theories in higher dimensions (*when supplemented* with dimensional reduction) may provide a promising framework for a deeper understanding of very high-energy physics. However, it is clear that the very fact of *imposing* the process of dimensional reduction in a given higher dimensional theory is equivalent to assuming a priori the dimension number 3 as a *natural* property of space, which is just what we are querying. To the best of our knowledge, there is as yet no satisfactory and unambiguous answer to the problem of dimensional reduction in the framework of these theories, even when the so called spontaneous compactification process is taken into account[2]. Thus we need to propose some physical argument to introduce another fundamental ingredient which, together with the former, will allow us to *start* the discussion of whether this number is indeed 3 -- but not necessarily to *determine* it. This ingredient will be provided

by the realization that a particular physical law is intimately dependent on the number of space dimensions. Historically, Kant's conjecture [2] that the three- -dimensionality of space may, in some way, be related to Newton's inverse square law has, indeed, opened a new way for the study of the problem of space dimensions. The main contribution of this conjecture to this problem is thus the suggestion that it can also be treated as a *physical* problem and does not belong exclusively to the domain of mathematics. It is relevant to stress here that, in spite of the importance of this conjecture, its physical support (if any) is yet to be understood.

Usually a third (and decisive) ingredient is always required to suggest a method which effectively connects the number of dimensions to some physical property. This is the most delicate part of any method one can propose for discussing the problem of spatial dimensions, which will be carefully examined throughout this paper. Here, only the physical aspects of this problem are discussed and, in particular, epistemological consequences of Ehrenfest's methodology aimed at fixing the number of space dimensions based on the so called "stability postulate" (see Section 2) are critically discussed. Some of the fundamental ideas related to the physical nature of this problem and to the question of the physical relevance of spatial dimension -- treated from different points of view [3] -- will also be briefly reviewed in Section 2 but, before discussing any principle that could be used to determine space dimensionality, we would like to say that we are convinced that it is impossible to disentangle questions concerning this subject from some kind of formalism representing a physical law. As Jammer put it [4]; "*... Hence it is clear that the structure*

of the space of physics is not,(...), anything given in nature or independent of human thought. It is a function of our conceptual scheme". This means that we accept that the physical concepts and the concept of reality itself acquire sense only within a theoretical construction where they can be discussed and realized. When the problem of space dimensions is considered, we must carefully examine the consequences of this fundamental point. Although this point has, in fact, motivated several works on the problem of spatial dimensions, it is in itself, at the same time, one of the main difficulties for the discussion of this problem, because the three-dimensionality of space is not questioned *a priori* when a physical law is established. This essential difficulty would be bypassed if we were able to prove the validity of the physical law in question whatever the number of spatial dimensions under consideration, rather than simply postulating it. The main aim of this paper is exactly to develop this point.

Concerning the origin of the results one may arrive at by discussing the *problem of the number of dimensions* in the way prescribed above, there is a straightforward and very important consequence we would like to emphazise, namely: *the mathematical structure of the formalism* one is considering (or simply a given physical equation(s)) is the *causa formalis* of the constraint obtained on the number of space dimensions. Actually we tend to consider this as the *unique* approach to start discussing the problem of the space dimensionality and this is essentially related to Jammer's idea recalled above. Thus this epistemological limitation seems to be inherent to this problem (so far as we understand it) and, in a certain sense, is well illustrated by Grassmann's words [5] : "*The*

concept of space can in no way be produced by thought, but always stands over against it as a given thing. He who tries to maintain the opposite must undertake the task of deducing the necessity of the three dimensions of space from the pure laws of thought, a task whose solution presents itself as impossible".

This paper is organized as follows. In Section 2 the present status of what we can learn from the formal extension of the number of space dimensions is discussed. Particular attention is given to Ehrenfest's and Weyl's contributions to this subject. A brief comment on the reality criterion associated with the "extra dimensions" in theories at higher dimensions is also presented is this Section 2. As a result of the criticism of the use of the "stability postulate", carried out in Section 3, an alternative approach to get at the proper dimensionality of space is presented in this same Section. In Section 4 it is shown how the task proposed in Section 3 can be carried out by considering a particular transition $R^3 \to R^n$ for the case of the black body phenomenology. This enables us to "demonstrate" the validity of the de Broglie relation for any R^n. This is the basis of Section 5, where thermal neutrons diffraction by crystals is presented as an example that completes the procedure proposed in Section 3. An upper limit for the dimensionality of space is therefore obtained. Some conclusions are drawn in Section 6.

2. *What one expects to learn from the transition* $R^3 \rightarrow R^n$.

As a first example, we can quote Ehrenfest's fundamental papers [6]. There, several physical phenomena, where qualitative differences between three-dimensional (R^3) and other n-dimensional (R^n) spaces were found, have been discussed. These aspects, which distinguish the R^3 physics from the R^n one, are called by him "singular aspects" and his works were aimed at stressing them. A crucial assumption is built in the main ideas contained in [6], namely, that it is possible to make the formal extension $R^3 \rightarrow R^n$ for a certain law of physics and, then, find one or more principles which, in conjunction with this law, can be used to single out the proper dimensionality of space. For this approach to be carried out, in general, the form of a differential equation -- which usually describes a physical phenomenon in a three-dimensional space -- is mantained and its validity for an arbitrary number of dimensions is *postulated*. For example, the Newtonian gravitational potential for an R^n-space, $V(r) \propto r^{2-n}$, is the solution of the Laplace-Poisson equation, $\sum_{i=1}^{n} \partial^2 V/\partial x_i^2 = k\rho$, in an n-dimensional space. Based on this general solution, Ehrenfest has used the postulate of the stability of orbital motion under central forces to get at the proper number of dimensions. In Ehrenfest's approach this additional postulate acts, therefore, as the *causa efficiens* of the dimensionality of space. It is just this part of his method the object of criticism in Section 3.

This general procedure is also followed in the work of Whitrow [7]. The importance of this approach was noted by Tangherlini [8] who proposed that, for the Newton-Kepler (N.K.) problem generalized to R^n space, the principle to determine the

spatial dimensionality could be summarized in the postulate that there should be stable bound state orbits or "states" for the equation of motion governing the interaction of bodies, treated as material points. This will be generically called from now on, the *stability postulate*. In his first paper [8.a] Tangherlini showed that the essential results of the Ehrenfest-Whitrow investigation are unchanged when Newton's gravitational theory is replaced by general relativity. In this same paper, the Schroedinger equation for the hydrogen atom in n dimensions is also considered. The above postulate, in conjunction with the assumption that the fields produced by the nucleus asymptotically approach a constant value at "large distances", gives n=3 in both cases. Thus the three-dimensionality of space discussed within the framework of Newtonian mechanics [6-7] or general relativity [8], and also quantum mechanics [8.a] (using a Coulombian potential), seems to be a result valid for a very large range of spatial scale -- we will turn to this point in Section 6. This briefly reviews how the "stability postulate" is used to throw some light on the problem of spatial dimensions.

From another point of view, these attempts based on stability arguments belong to a class of arguments epistemologically different from that contained in the work of Weyl [9], which we shall briefly review here. His basic approach was to construct a new unified theory of gravitation and electromagnetism based on a gauge-invariant non-Riemannian geometry. In this scheme, Weyl pointed out that there is a strong relation between the metric structure of space-time and physical phenomena, which could lead to a deeper understanding of Maxwell's electromagnetic theory as well as of the four-dimensionality of space-time. Weyl showed that only in (3+1)-

dimensional space-times can Maxwell's theory be derived from a simple gauge-invariant integral form of the action, having a Lagrangean density which is conformally invariant. This could be considered as an example of how a set of physical phenomena, here synthesized by Maxwell's theory, could be used to impose some restrictions on the dimensionality of space[3]. The structure of Maxwell equations and the gauge principle are, respectively, the *causa formalis* and the *causa efficiens* of the four-dimensionality of space-time. The two essentially different (although complementary) features of Ehrenfest's and Weyl's methodology can be summarized as the difference between the two following questions: (i) How does it become manifest in the fundamental laws of physics that space has three-dimensions, and (ii) How do the fundamental laws of physics entail spatial dimensionality? All work based on the "stability postulate" hinges on the former question because the constraint on n is reached as a consequence of a "singular aspect" of a physical law that is *supposed to be still valid* under the transition $R^3 \to R^n$. The latter is implicit in Weyl's work where the structure of Maxwell theory *cannot* be mantained[4] if $n \neq 3$. The second question can be well illustrated by the concluding paragraph of Tangherlini's paper [8.a], where he says: *with further work, we may come to regard n=3 as an eigenvalue.*

However, even from a classical point of view, Weyl's demonstration of the four-dimensionality of space-time is not complete: the gravitational law should also be derivable from the same requirements of invariance as for electromagnetism. The point is that although Weyl's unified theory is a good place for giving an answer to the problem of spatial dimensions, it should be mentioned that this theory has been criticized in the

literature [10]. In any case, in order to consider complete such kind of demonstration, today we must clearly take into accont also *strong* and *weak* interactions. We will turn later to this point at the end of this Section.

Other attempts to create a geometry in which the gravitational and electromagnetic potentials together would determine the structure of space were carried out. An example is Kaluza-Klein theory [11] -- which is presently enjoying a great revival of popularity in connection with the modern theories of supergravity -- where the number of components of the metric tensor was increased by changing the number of spatial dimensions. A fifth dimension was added to the usual four dimensions of physical space-time. In the work of Kaluza, the a *priori* four-dimensional character of the physical world is assumed when the author looks for a suitable choice of coordinates, in such a way that the components of the metric tensor be independent of the fifth coordinate. In other words, this coordinate has no direct physical significance. Thus it is quite clear that this kind of approach to a unification program could not lead to a satisfactory answer to the problem of spatial dimensions[2]. However it should be said that an argument aimed at showing that a necessary condition for a unified field-theoretic description of gravity and electromagnetism implies that the world be four dimensional, as discussed by Penney [12]. The four dimensionality of space-time is also required by Schönberg's [13]. In this interesting work an electromagnetic foundation for the geometry of the world-manifold is proposed. Einstein's gravitational equation appears as complementing the set of Maxwell equations, giving rise to a natural fusion of the electromagnetic and gravitational theory. The electromagnetic

theory is formulated in a differentiable manifold devoid of any metric and affine structure. In this formulation there is no *a priori* relation between $F_{\mu\nu}$ and $F^{*}_{\mu\nu}$, involved in the homogeneous and non-homogeneous Maxwell equations, respectively. The foundation of the four-dimensionality of the world-manifold (space-time) is given by the structure of the Maxwell equations in terms of the two basic tensors $F_{\mu\nu}$ and $F^{*}_{\mu\nu}$, which are both antisymmetric covariant of the same order. It is important to stress that, in this approach, the four-dimensionality of the space-time is essentially associated to the differential electromagnetic equations, without any consideration about the relation between $F_{\mu\nu}$ and $F^{*}_{\mu\nu}$ and without requiring a metric-space as in Weyl's work.

There are other attempts to unify not only electromagnetic and gravitational forces but all the fundamental forces, considering space-time with a high number of dimensions as,for example, supergravity or the ten-dimensional space-time superstring theory [14]. But, whenever the problem of space-time dimensionality is considered in the framework of these (super-) theories, we face the problem of the physical reality of these "extra" dimensions. Independently of any particular theory, as pointed out by Mansouri and Witten [15], *if we wish to take the physical existence of the extra dimensions seriously, we must develop a systematic method for studying the effects of the extra dimensions (...) Since there is no evidence for the existence of the extra dimensions at the shortest distance which can be probed at present, it (any such method) must explain how this can be attributed to some intrinsic property of a higher dimensional theory. It must (also) provide a quantitative method for* studying *the consequences of the dependence on the extra*

dimensions. Complementing this picture we can always ask whether the ten-dimensional superstring theory, for example, can tell us, in a straightforward and unambiguous way, that we are living in a ("almost flat") four dimensional space-time. The fundamental question is; *why* dimensional reduction? Up to now, the answer to this question, i.e., the four-dimensionality of the physical world-manifold is yet, in the last analysis, an *ad hoc* ingredient in these theories.

On the other hand, it was shown [3.d] that only for space-time dimensionality greater than four, the fundamental constants of electromagnetism (e), quantum theory ($\hbar$), gravity (G) and relativity (c) are all included in a single dimensionless constant -- which should have, in a unified theory, a similar rôle to that played by the Sommerfeld constant $e^2/\hbar c$ in the quantum electrodynamic theory. Thus the apparent necessity of going to a high dimensional space-time , in order to carry out the unification program, brings with itself the problem of how to explain all the well-known phenomenological manifestations of the four-dimensionality of space-time in the framework of this new theory, and the question of the reality of the "extra dimensions": both are clearly still open questions in physics.

3. *Criticism of the use of the stability postulate.*

We can ask if the "stability postulate" -- applied to the N.K. problem or hydrogen atom -- is actually a good choice for deriving the spatial dimensionality or not; or more specifically, if we can really *prove* that n=3. We understand that the use of this postulate enables us only to exclude the possibility of having a class of natural phenomena in a space other then our own, with an arbitrary dimension, as pointed out by Poincaré [16]. Then, when we consider the example of the hydrogen atom, as described in Section 2, the results obtained from that postulate must be stated as follows: there is no $\mathbb{R}^n$ *other than* $\mathbb{R}^3$ where the phenomenon under study is described by a generalized Schröedinger equation that has the same form as in the case n=3, and whose solution is *also* stable -- and that is all. Indeed, when Ehrenfest used the Bohr atomic model for the hydrogen atom, the stability of matter in three dimensions was *already* assured by the postulate of angular momentum quantization, and this justifies the term *also* underlined above. The fact is that he could not have used Rutherford's model -- which is clearly unstable in $\mathbb{R}^3$ -- plus the stability postulate to derive the number of dimensions as being just 3. Thus n=3 is *a priori* favoured in this case. Apart this feature, it is clear that it is only when the formalism, previously generalized to an n-dimensional space, presents a singular behaviour under this generalization, that the "stability postulate" can be used as a method to fix the proper dimensionality of space. The range of applicability of the "stability postulate" is therefore strongly restricted to a very particular class of formalisms. Moreover, these two intrinsic

characteristics of this method clearly do not solve the essential difficulty discussed in Section 1 and, from the epistemological point of view, show that the use of the "stability postulate" to fix n is not satisfactory.

We can now ask if we cannot imagine a phenomenon or a physical state that could only be stable in an R^n with $n>3$, but described by an equation having the same form as in R^3, and analyze the consequences of this assumption. For example, we can ask why we do not observe in a Stern-Gerlach [17] experiment the dissociation of a beam of spin 1/2 particles in more than two lines. Or, in other words, is the stability of these particles (e.g. electrons), described by a Dirac-equation, a manifestation of a particular space dimensionality? Particles having higher spin must be unstable in R^3, while stable in some $R^n \neq R^3$ and so, having a mean life time so small in three dimensions this kind of experiment could not be carried out. This conjecture could indicate that if the "stability postulate" were applied to the evolution of a massive spin 3/2 particle, described by a (hypothetical) Dirac-like equation, the number of spatial dimensions derived could be greater than 3! This is an example where the results obtained by using the "stability postulate" do not depend on the form of the equation but, instead, on what kind of object this equation describes.

The alternative principle we want to propose may be stated as follows: "Given a formalism in a certain dimension, (usually three) we must, based upon its fundamental equations, ask whether other forms (or equations) are valid in a higher n-dimensional space for *all* n, rather than simply postulating the validity of the same formalism in a different dimension". In other words we shall not be concerned only with formalisms which

are singular in a certain n (usually three). On the contrary, we shall look for situations which do not present those singularities. Then this alternative principle could be used to discuss the spatial dimensionality (Section 4). It will certainly describe several phenomena and their observability could be used for that purpose. It is clear, however, that, in this case, the constraints obtained will be weaker than those obtained when the "stability postulate", or the search for singular aspects of the transition $R^3 \to R^n$, are considered. Nevertheless, this procedure has the advantage that we can guarantee *a priori* that the fundamental law, used to describe a certain kind of phenomenon, is valid for *any* R^n, which is not possible in other procedures as pointed out in Section 1. Then, when this alternative procedure is applied we can conclude: the dimensionality of space *is* a number included in a certain range -- 3 need not *a priori* be favoured.

4. *Black body phenomenology: a non singular aspect of the transition* $\mathbb{R}^3 \to \mathbb{R}^n$.

There are several physical laws in which the dimensionality of space affects the results, but the transition $\mathbb{R}^3 \to \mathbb{R}^n$ does not have a "singular" behaviour, and thus these laws were not discussed in the works of Ehrenfest [6]. An example is Wien's law which, in its generalized form[5], becomes $\rho = \nu^n F(\nu/T)$. However, we would like to point out that, although this transition has no "singularity", the black body phenomenology extended to $\mathbb{R}^n$ contains an important feature that must be emphasized. Indeed we can use it in order to "demonstrate" the validity of the de Broglie relation in other $\mathbb{R}^n$, as will be shown now.

If we assume Planck's energy quantization to determine the explicit form of the function F, we still find that the energy of a quantum is $\epsilon_o = h\nu$, for any $\mathbb{R}^n$. This is easily seen if we remember that the energy eigenvalue of the Schroedinger equation for the harmonic oscillator gives Planck's result up to the ground state energy. The transition to $\mathbb{R}^n$ only changes this energy value from $3h\nu/2$ to $nh\nu/2$, and then Planck's hypothesis is still valid, i.e., the quantum energy is proportional to the *first* power of the frequency ν. We note that this result clearly depends on the classical potential energy $V = kx^2/2$ used in the Schroedinger equation, and a brief digression about it is necessary.

When a spring is displaced from the equilibrium position, we learn from the experiment that, for small displacements, the restoring force is proportional to the displacement, and that is all. It does not matter in which

direction the displacement takes place and the problem can be called a quasi one-dimensional problem. The result is the same in one, two or three dimensions and this is quite different from the Newtonian-Keplerian potential, for which a qualitative difference among R^3 and R^2 exist [6]. Thus we can expect that the form of Hooke's potential could be the same for all R^n. However, even if this is not true, but if the generalized potential has a minimum, we can always approximate it by the harmonic potential, in the case of small oscillations, whatever R^n is considered (a particular case of Morse theorem). After this note, we can turn back to the original problem.

We can still assume that the energy trapped in a cavity (a model for a black body) corresponds to the energy of a collection of "photons" which must satisfy Einstein's relation $M^2 = g_{\mu\nu} p^\mu p^\nu$, generalized to R^n -- it is the same kind of generalization made for the potential energy, where only the number of components of the metric (the scalar product) was increased. By imposing that a quantum must also satisfy the above relation, it follows immediately that the de Broglie relation $\lambda = h/p$ is valid in *any* R^n, because Planck's quantization law did not change. Thus we can also conclude that, as the de Broglie relation is exact in *any* R^n, the momentum p of the particle cannot be a function of its coordinate x, and so we should expect that Heisenberg's uncertainty relations are also valid [18]. This result, in a certain sense, properly supports the initial generalization of the Schroedinger equation, as it should be expected that the equivalence between Heisenberg's and Schroedinger's pictures must be mantained for other R^n. This feature is a self-consistency test for this generalization, which, to our knowledge, has not been used in

the past literature.

So, it has been shown in this Section that even though the transition $R^3 \to R^n$ does not show "singular" aspects, there is a case in which we can still perform it (justified *a posteriori*) and conclude something about the validity of other physical law in R^n. The advantage of this procedure was already discussed in Section 3.

We will now apply the result of this Section to a particular physical effect -- the possibility of having thermal neutron diffraction by crystals in an R^n space. It is essentially explained by the de Broglie hypothesis and then an upper limit for spatial dimensionality will be obtained, based on the general arguments presented in Section 3.

5. *Thermal neutron diffraction by crystals as a means for obtaining an upper limit for the spatial dimensionality.*

It is well known that a thermal neutron beam falling onto a crystal lattice gives rise to diffraction phenomena [19] -- known as "neutron diffraction" -- analogous to those observed when we use incident X-ray beams. The passage of thermal neutrons through matter gives rise to scattering processes which are most readily understood in terms of the wave properties of the neutrons [20]. We define as "thermal" a neutron whose kinetic energy corresponds to the mean energy of thermal agitation at temperature T. Usually we can write $p^2/2m \simeq 3K_B T/2$, where the factor 3 arises when we consider R^3-space and only 3 degrees of freedom, corresponding to translational motion, are assumed for the neutron, i.e., by hypothesis, one does not take into account any internal degree of freedom. Therefore, if we assume the energy equipartition theorem to be still valid for an R^n-space, each degree of freedom will contribute with $K_B T/2$ and the factor 3 should be replaced by[6] n.

Since the classical thermodynamic laws do not show singular aspects concerning the $R^3 \rightarrow R^n$ transition[7] it is still possible to thermalize a neutron beam in an R^n-space. Thus the de Broglie wavelength associated to the neutron is $\lambda = h/p$, where $p \simeq (nmK_B T)^{1/2}$. From now on λ will be considered as a function of both the dimensionality of space and neutron velocity ("temperature"), with n being a parameter to be determined. The starting point is, therefore, that neutron thermalization may occur in an n-dimensional space. The subsequent process - neutron diffraction -- simply acts as the *detection* of something that has happened inside a nuclear

reactor, for example. To measure λ use will be made of Bragg's law [21].

If d represents the grating spacing the condition for coherent reflection is given by Bragg's law

$$2d \sin\theta = \ell\lambda \quad , \ \ell = 1,2,3,...$$

For diffraction patterns to be observed, the wavelength must be of the order of magnitude of the mean distance between crystallographic Bragg planes (which in R^3-space are given by the so called Miller indexes [22], easily generalized to R^n), but *can not exceed* 2d. In this case Bragg's law has no solution for integer ℓ and there is no diffraction pattern.

The distance d can be measured by using X-ray techniques and thus is independent of the dimensionality of space, i.e., for X-rays the relation $p \approx (nmK_B T)^{1/2}$, valid for massive particles (as neutrons, helium atoms, hydrogen molecules, etc.), is obviously not valid any longer.

We can then conclude that, in an R^n-space, diffraction gratings [23] do exist -- the spacing grating being independent of n -- and it is possible to thermalize a neutron beam. However it is still possible to "measure" n even in the limiting case where a "one-dimensional crystal" is used as a "rod" because λ is, by definition, "one-dimensional" and the knowledge of n comes through the measure of λ. Thus the above requirement that diffraction gratings exist in R^n seems to be superfluous. In any case, the 3-dimensionality of the macroscopic crystal does not necessarily say anything about the space dimensionality of the microscopic characteristic length of thermal neutron production. This information is carried out by the neutron and will be

revealed by the crystal lattice. To make the point, we are taking into account the possibility that the space dimensionality may be dependent on the spatial scale (or energy scale) we are probing.

In its application to solid state problems, neutron diffraction is similar in theory and experiment to X-ray diffraction but, in fact, regarding some particular aspects, they could be considered as two complementary techniques [20]. The experimental apparatus we will consider consists of a monochromatic neutron beam (obtained with usual techniques [20.b,c]) and a crystal. The mean distances between the Bragg's planes are measured by using X-ray techniques. Given a certain crystal one tries to determine the larger value of these distances which, in general, lies on an axis of symmetry of the crystal. The neutron beam is then sent on the crystal in such a way that it will be diffracted by the parallel planes having as relative distance the aforementioned value. The advantage of this procedure will be soon understood.

It is well knowm from optics that, even when the number of slits in a diffraction grating is not very large, the intensity of secondary maxima in the diffraction pattern is much reduced, compared with the intensity of principal maxima [23]. In the case of neutron diffraction by a crystal one has a very large number of "slits" -- the mean intervals between atoms -- which, clearly, renders difficult the experimental observation of a high order spectrum. But this does not mean that they could not be observed in principle. From Bragg's law it follows that to have a second order spectrum we must have $\lambda \leqslant d$; for a third order one we need $\lambda \leqslant 2d/3$, and so on. The condition for having a diffraction pattern with *only* the ℓ-th. order spectrum is,

therefore, $2/(\ell+1) \leqslant \lambda/d \leqslant 2/\ell$. The possible ranges for the neutron wavelength are then always different and this is an important point, as we will see now.

Suppose one can vary (increase) the number of spatial dimensions for a given constant temperature; for example from n=3 to n=12. As λ is proportional to $1/\sqrt{n}$, this corresponds to dividing the wavelength by a factor 2 and, therefore, it is equivalent to going from a spectrum of order ℓ to one of order 2ℓ. This fact, naturally, strongly suggests that one should observe the *first* order spectrum, as far as one is looking for an upper limit for n. We can, thus, perform a *gedanken* experiment where it is possible to prepare a monochromatic neutron beam satisfying the condition $d \lesssim \lambda \lesssim 2d$, by varying the neutron velocity and, consequently, λ, which assures us that no higher order spectra are present in the diffraction pattern other than the first one. Only if one can change λ by a factor 2 and still have the same order diffraction pattern is one sure that it is the first order spectrum that is observed, because we must remember that we are taking n as an unknown quantity. After being sure that this is the case, we can then use the relation $\lambda = h(nmK_B T)^{-1/2}$ for determining n. Therefore, we can conclude that the observation of thermal neutrons diffraction, under the condition $d \leqslant \lambda \leqslant 2d$, can be used to measure[8] n.

We shall now analyse the available experimental data. It is known from X-ray measurements that a typical value for d in crystalline solid is $d \gtrsim 10^{-10}$ m and the characteristic temperature is $T \simeq 300$ K. For neutron beams, from what has been said above, both values are independent of space dimensionality. This is the fundamental fact that allows us to use $d \lesssim \lambda \leqslant 2d$, which gives us the approximate limit $n \lesssim 5$. For a fixed value of

the temperature, one may ask whether a particular crystal whose d value is such as to test n=3 does exist.

The wave aspect of the phenomenon discussed in this Section might lead to supplementary restrictions on the value of n.

In classical physics, diffraction effects can be explained on the basis of a wave theory by the application of Huygens' construction together with the principle of interference. In $\mathbb{R}^{2n}$-space it is well known that Huygens' principle does not hold [24]. It should also be noted that Hadamard [25] has shown that the transmission of wave impulses in a reverberation-free fashion is possible only in space with an *odd* number of spatial dimensions[9] and, in these cases, Huygens' principle is valid for single differential equations of second order with constant coefficients. However, Hadamard's conjecture states that this theorem holds even if the coefficients are not constant [26]. The Huygens' principle is then expected to be valid in any $\mathbb{R}^{n}$-space where n is odd. Now we shall assume that the classical results discussed in this paragraph remain valid when we consider the diffraction of matter by crystals -- traditionally explained by de Broglie's hypothesis within quantum mechanics[10]. This point is far from trivial and is now under investigation. The difficulty comes from the fact that Hadamard's results apply to d'Alembert equation, of hyperbolic type, while Schroedinger equation is parabolic. So, within the above assumption, we can conclude that thermal neutron diffraction gives an upper limit for the spatial dimensionality which is an odd integer less than or of the order of five.

We hope the *gedanken* experiment performed here may, in practice, be carried out in the near future.

6. Concluding remarks.

In this paper we have discussed the validity of applying the "stability postulate" to the problem of spatial dimensions. It was shown that this kind of approach naturally favours *a priori* n=3. An alternative approach is proposed where, basically, it is suggested that one must first demonstrate that the ultimate law used to derive spatial dimensionality is valid in a generic R^n, rather than simply postulating the validity of the same equation for an arbitrary R^n. From this approach one finds that the constraints obtained on the spatial dimensions are not only weaker (upper limits) than those obtained by using stability arguments, but have also a different nature, which we consider more appropriate to this problem. The main advantage of our methodology is that it is able to bypass an essential difficulty inherent to the problem of the number of spatial dimensions, namely: n=3 is never questioned *a priori* when a physical law is established. Clearly it is not our scope to deduce the number of dimensions of space from a purely conceptual law [5], but provide a constructive scheme to get at it. As stated in the Introduction, we believe that the structure of physical space -- in particular its dimensionality -- is a function of our conceptual scheme but it does not seem possible to deduce the spatial dimensionality from it. In the last analysis, one should resort to phenomenology to determine it.

In this paper, the fundamental equations generalized to R^n were the Schröedinger equation and the Einstein energy-mass relation. The validity of the de Broglie relation for any R^n properly supports the initial generalization of the Schröedinger

equation (Section 4) and, at the same time, gives a justification for it, in general not found in other cases. Then, using this result, we have suggested the phenomenon of thermal neutron diffraction by crystals as a means to determine the number of spatial dimensions. As a consequence, we have found an upper limit for n, which is an odd integer (by assumption) less than or of the order of five. We consider the *gedanken* experiment performed in Section 5 an *experimentum crucis* for the problem of spatial dimensions and hope it may, in practice, be carried out in the near future.

Let us now make some comments about the nature of the different approaches, concerning the physical problem of spatial dimensions, quoted in this paper. We can divide them into two distinct clases. The first one corresponds to topological arguments: to this class belong Whitrow's bio-topological argument [7] and Poincaré's argument, based on the *analysis situs* [3.a,16]. The kind of constraints obtained from it is a lower limit for spatial dimensionality, e.g., $n \geq 3$. In the second class, we group all other arguments where it is necessary to introduce a metric space and this seems to restrict the range of possible values of n. A metric space is introduced whenever we consider the existence of an interacting system as the starting-point in the discussion of the problem of spatial dimensions. It is clear that to begin with an interacting system, knowledge of the form of the interaction -- the physical law describing the phenomenon in a space-time manifold -- is a necessary condition. This renders the class of "metric arguments" more "complete" *a priori*, in the sense that it contains more information than the class of "purely topological arguments". The difference can be considered as the cause of the

difference between the two types (or classes) of constraints for n. There is, however, an exception to this general picture that should be emphasized: the Maxwell electromagnetic theory. We would like to point out here its distinguished rôle in the physical problem of spatial dimensions. All the attempts to obtain the space dimensionality which are based upon the structure of Maxwell's equations (no matter whether they belong to the class of metric approaches or not) give n=3.

It is not perhaps out of place to present now some almost obvious remarks about time (and space) "scale" of the arguments previously discussed. Ehrenfest's stability argument is valid for distances of the order of the solar system and in a time scale large enough to make the evolution of life possible on Earth (as noticed by Whitrow[11]). Tangherlini's work about the stability of H atoms can be invoked here to suggest the validity of chemistry in the same time scale as a necessary, although not sufficient, condition -- at least chemical thermodynamics of irreversible process should be also valid). The presence of atomic spectra in remote stars may also indicate that space has had the same dimensionality at cosmic scale. To have such a cosmic constraint on space dimensionality is very interesting and we hope to treat this poit in a future communication.

It is also interesting to note that all the arguments presented up to now that depend on the presence of matter are essentially metric. This is the case of Ehrenfest-Tangherlini--Whitrow. Topological arguments are basically related to the idea of a field -- this is the case of Maxwell's theory, as mentioned before, and Wien's law, which involves, essentially, the equilibrium of radiation.

As for most physical arguments used to obtain the spatial dimensionality it is necessary to introduce a metric space, our two last critical comments are dedicated to clarify some aspects involving it.

Firstly, in ref. [8.a] the author was led to conclude that the stability postulate, applied to the N.K. problem, fixes the dimensionality of space and, at the same time, is an absolute prerequisite for a comparison of relative distances between bodies to be physically possible. However, taking into account the analysis we have done and the example we have proposed in the preceding Sections, we are led to conclude that, in fact, it is the physical interaction between two bodies, or two systems, that necessarily leads to the introduction of a metric-space in order to be able to obtain the number of spatial dimensions in these cases; but neither the stability postulate nor a metric-space [13] are indeed *necessary* to fix the dimensionality of space.

Secondly, we would like to point out that the necessity to have a metric-space for most physical arguments concerning the problem of spatial dimensions, brings whith itself the notion of distance, traditionally based on the differential homogeneous quadratic form, $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, which, in the last analysis, is an arbitrary choice -- indeed there is no logical argument for excluding other forms for the line element as ds^4, ds^6, ds^8, etc. In spite of this (up to now) logical impossibility the importance of investigating the nature of the exponent 2 was emphasized in an early work by Ehrenfest [6.b]. His conjecture that this 2 could be related to the dimensionality of space is, however, yet to be demonstrated. Nevertheless, so far as the formula for a line element in a

manifold of n dimensions is viewed as arbitrary, some care must clearly be exercised in advancing Ehrenfest's conjecture. If, on the contrary, this conjecture is shown to be actually true, we can ask whether it can be related in some way to Fermat's last theorem.

In conclusion, we would like to say that although some epistemological difficulties concerning the use of "stability arguments" are bypassed by the methodology proposed in this paper, there remains, somehow, a certain incompleteness since a physical event takes place not only in space, but in *space-time*. Thus the problem of the number of space dimensions and that of time dimensions are probably not independent. One can then ask whether it is possible to propose a more general methodology which could be able to constrain not only the number of spatial dimensions but also, simultaneously, time dimensionality. Are these numbers actually related? Is it possible to prove time to be one-dimensional by disclosing space dimensionality and/or vice-versa? It is our conviction that, in the future, further efforts should be made trying to answer these questions, whether or not a deeper comprehension on the problem of space dimensionality is to be reached.

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FOOTNOTES:

1 - An example of criticism where the three-dimensionality of space is considered as a contingent feature can be found, for example, in MACH E. , *Die Mechanik in ihrer Entwick*lung *historisch-kritisch dargestellt*, Leipzig, 1883 , Italian transl., *La meccanica nel suo sviluppo storico-critico*, Torino, Boringhieri 1977, pp. 479-80.

2 - It is shown that spontaneous dimensional reduction in any Kaluza-Klein theory always yields a compactified extra space. However, without an adjustable cosmological constant, the scale of the ordinary four-dimensional space-time is the same order of magnitude as that of the compactified space. Cf. TOSA Y., *Phys. Rev.* D30 (1984) 339; See also CREMMER E. and SCHERK J., *Nucl. Phys.* B108, (1976), 409. Now, for illustrating the present difficulties on this subject, concerning superstring theory, we can quote Ferrara's words: *Superstrings are 10-dimensional theories of one-dimensional extended objects, so their relation to the physical world is only possible if they undergo a mechanism of spontaneous compactification from D=10 to D=4 dimensions. The study of spontaneous compactification of the fully fledged superstring theory is a formidable task to achieve, since it requires the knowledge of the full second-quantized version of the interacting theory.* Cf. FERRARA S. "Matter Coupling in Supergravity", in *Superstrings and Supergravity - Proceedings of the Twenty-Eighth Scottish Univ. Summer School in Physics*, A.T. DAVIES and D.G. SUTHERLAND eds., Oxford, Univ. Printing House, 1985, p.381.

3 - Indeed, this result is based on classical arguments and one can argue that this is not the only example. In fact, one gets the same constraint on n when extending Weyl's approach to classical Yang-Mills theory -- YANG C.N. and MILLS R.L., *Phys. Rev.* 96, (1954) 191.

4 - *What inner peculiarities distinguish the case n=3 among all others? If God, in creating the world, chose to make space 3-dimensional, can a reasonable explanation of this fact be given by disclosing such peculiarities?*, cf. WEYL H., in *Philosophy of Mathematics and Natural Science*, revised and augmented english transl., Princeton, Pinceton Univ. Press, 1949, p. 70. Weyl has shown that electromagnetism plays such a particular rôle; cf. *ibid.* pp. 136-37 and ref. [9].

5 - It should be noted that this generalization follows purely from the validity of thermodynamics in $\mathbb{R}^n$, leaving the explicit form of $F(\nu/T)$ open. See also footnote 7.

6 - Here we are identifying the space dimensionality with its number of translational degrees of freedom.

7 - Assuming time to be one-dimensional (as always assumed in this work) and "flowing" in a definite direction. However, the statement made in the text seems to be no longer true if one tries to develop a thermodynamical theory in the framework of general relativity. Cf. STUECKELBERG E.C.G., *Helv. Phys. Acta* 26, (1953), 417; STUECKELBERG E.C.G. and WANDERS G., *ibid* 26, (1953), 307. We thank Dr. M.O. CALVÃO for pointing out to us these refs.

8 - Another recent proposal for measuring the number of dimensions of space-time, which leads to a fractional dimension, can be found in; ZEILINGER A. and SVOZIL K., *Phys. Rev. Lett.* 54, (1985), 2553. Some consequences of a modification of Newton's and Coulomb's laws, introduced by assuming a non integer value for the spatial number of dimensions, are examined in; JARLSKOG C. and YNDURAIN F.J., *Europhys. Lett.* 1, (1986) 51. There, it is inquired how large can the deviations from the "standard" n=3 value be. Also the recent work by GRASI A., SIRONI G. and STRINI G., *preprint of Phys. Dept. of Milano Univ.* (to appear in *Astr. Space Sci.*) is aimed at setting upper limits to such deviations. It should also be mentioned that in a recent paper of GASPERINI M., *preprint No. DFTT 86/24 of Phys. Dept. of Torino Univ.* (to appear in *Phys. Lett. B*) it is shown that the modification of Newtonian potential -- deviation from the 1/r gravitational potential - following from a deviation of the number of spatial dimensions from the integer value of 3, can *also* be obtained in the usual four-dimensional context, provided that the SO(3,1) gauge symmetry of gravity is broken. Thus this result gives rise to the possibility of ambiguous interpretations for small deviations of the Newtonian gravitational law, but does not affect Coulomb's law.

9 - Further, for the transmission of a wave signal to be free of distortion it can be shown that n=1 and n=3 are the only possibilities.

10 - Furthermore Bragg's law has been obtained in an alternative way, without using matter waves and, therefore, independently of Huygens' construction. Indeed, it has been argued by BUSH R.T.,

in *Lett. Nuovo Cimento* 44, (1985) 683 and also *ibid* 34, (1982) 363; *idem* 36, (1983) 241, that a direct particle interpretation based on Heisenberg's uncertainty principle can be given to the interference pattern produced by a regular grating.

11 - One may add the following remark to Whitrow's argument about this subject [7.b]. It is not sufficient that the intensity of solar radiation on Earth's surface should not have fluctuated greatly for still having life on Earth. The fact that Sun's spectra of radiation did not fluctuated very much is also required.

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