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ISOMORPHISM BETWEEN MATRICES AND QUATERNIONS

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ABSTRACT

It is shown that the correspondence matrix-quaternion is such that to each physical equation there corresponds a set of equivalent matrix equations.

Key-words: Matrices; Quaternions.

The isomorphism between the algebra of quaternions over the field of complex numbers and the algebra of real (8x8)-matrices was recently emphasized by Jolly[1]. The mapping from the quaternionic into matrix representation was accomplished by using a matrix Q. As an application Maxwell's equations were written in matrix form.

It is the purpose of this note to show that the matrix Q can be written in a quaternionic form. Matrix equations are nothing but quaternion equations. Moreover, it will be shown that, to each physical equation, there corresponds a set of equivalent matrix equations.

The general form of a biquaternion $\overline{Q}$ is

$$\overline{Q} = (a+ei)e_0 + (b+fi)e_1 + (c+gi)e_2 + (d+hi)e_3 \quad , \tag{1}$$

where a, b, c, d, e, f, g and h are real numbers

$$e_0^2 = 1 \quad , \quad e_j e_k = -\delta_{jk}+\varepsilon_{jkr}e_r \quad (j,k,r = 1,2,3) \quad . \tag{2}$$

The matrix Q corresponding to this biquaternion, given by Jolly in a more compact form, is[2]

$$Q = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \tag{3}$$

If the following (4x4)-matrices

$$\Gamma_1 = \begin{bmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{bmatrix} \quad , \quad \Gamma_2 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \quad , \quad \Gamma_3 = \begin{bmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{bmatrix} , \tag{4}$$

are introduced, it is possible to write

$$A = aI + b\Gamma_1 + c\Gamma_2 + d\Gamma_3 \quad , \tag{5a}$$

and

$$B = eI + f\Gamma_1 + g\Gamma_2 + h\Gamma_3 \quad . \tag{5b}$$

From (4) it is seen that

$$\Gamma_j\Gamma_k = -\delta_{jk} + \varepsilon_{jkr}\Gamma_r \quad (j,k,r = 1,2,3) \quad . \tag{6}$$

In view of (3), (5a) and (5b) the matrix Q can be written as

$$Q = (a+Je)\bar{I} + (b+Jf)\bar{\Gamma}_1 + (c+Jg)\bar{\Gamma}_2 + (d+Jh)\bar{\Gamma}_3 \quad , \tag{7}$$

where $\bar{I}$, J, $\bar{\Gamma}_1$, $\bar{\Gamma}_2$ and $\bar{\Gamma}_3$ are (8x8)-matrices defined as follows

$$\bar{I} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} , \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} , \quad \bar{\Gamma}_k = \begin{bmatrix} \Gamma_k & 0 \\ 0 & \Gamma_k \end{bmatrix} \quad k=(1,2,3) \tag{8}$$

Here I is the (4x4) unit matrix and Γ_k are defined by (4). The matrix J corresponds to the imaginary quantity <u>i</u>, whereas $\bar{\Gamma}_k$ correspond to the quantities e_k of the relation (1).

From (8) it is seen that $\bar{\Gamma}_k$ can have several realizations. Each realization corresponds, of course, to a different matrix Q in (7). In (4) one realization of Γ_k is presented. Another one is obtained by using the following set of Γ'_k matrices

$$\Gamma_1' = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \quad , \quad \Gamma_2' = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} \quad , \quad \Gamma_3' = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} \quad , \qquad (9)$$

that satisfy (6). Two more sets are obtained by inserting σ_1 and σ_3 in (4) instead of σ_2.

It should be noted that I, $\overline{\Gamma}_1$, $\overline{\Gamma}_2$, $\overline{\Gamma}_3$, J, $J\overline{\Gamma}_1$, $J\overline{\Gamma}_2$ and $J\overline{\Gamma}_3$ form a group. The sets (9) and (4) are equivalent. One has, indeed,

$$S\Gamma_k S^{-1} = \Gamma_k' \quad , \qquad (10)$$

where

$$S = \begin{pmatrix} I & i\sigma_2 \\ -\sigma_3 & \sigma_1 \end{pmatrix} \quad . \qquad (11)$$

In general it is not true that

$$SJS^{-1} = J \quad .$$

In the aforementioned example one has

$$SJS^{-1} = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} \neq \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad .$$

Nevertheless, the matrix $\overline{J}$, which satisfy the relation

$$SJS^{-1} = \overline{J} \quad ,$$

is such that

$$\overline{J}^2 = -I \quad ,$$

if (8) holds. Therefore, matrices Q_Γ and $Q_{\Gamma'}$ defined as (7) with the help of matrices such as (4) and (9), which are equivalent, satisfy

$$SQ_\Gamma S^{-1} = Q_{\Gamma'} \quad .$$

In general equations like

$$DF = (4\pi/c)\ J \quad ,$$

given by Jolly are invariant under the transformations

$$D \rightarrow S^{-1}DS \quad , \quad F \rightarrow S^{-1}FS \quad , \quad J \rightarrow S^{-1}JS \quad ,$$

so that to each physical equation there corresponds a set of equivalent matrix equations.

REFERENCE

[1] D.C. Jolly, Lett. Nuovo Cimento, 39, 185(1984).

[2] There are four misprints in the Q matrix presented by Jolly.