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GENERAL AND UNIFIED SOLUTION FOR PERFECT-FLUID HOMOGENEOUS AND ISOTROPIC COSMOLOGICAL MODELS

by

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ABSTRACT

All perfect fluid spatially homogeneous and isotropic cosmological models (without cosmological constant) are solved in general and unified form when an equation of state $p = (\gamma-1)\rho$ is assumed. The explicit dependence of cosmological time on the metrical scale factor is determined for any values of γ and spatial curvature parameter ε. A set of four infinite numerable sequences for values of γ, all of them consistent with the energy conditions and each starting from one of the values 0, 1/3, 1, 4/3, includes all cases having solutions described by elementary functions. A generation technique yields the construction of all solutions in each sequence. By the use of the conformal time coordinate, the differential equation for the scale factor may be set in the form of that describing the classical motion of a particle subject to a linear force. Closed models are analogous to harmonic oscillators and their "half periods" are determined as an explicit function of γ, both for the conformal and cosmological times.

Key-words: Relativistic cosmology; FRW models; Perfect fluid.

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1 INTRODUCTION

The system of Einstein's field equations for spatially homogeneous and isotropic cosmological models demands, to be solved, a choice for the values of two parameters. They may formally be taken as continuous and their origin is twofold:

a) The spatial curvature parameter ϵ, which may be positive, null or negative, completely characterizes the admissible homogeneous and isotropic geometries described by the Robertson--Walker[1,2] line element.

b) The energy momentum tensor taken is in the form of that of a perfect fluid, described by its energy density ρ and isotropic pressure p. The problem is then reduced to the solution of a couple of field equations expressing ρ, p and the scale factor as functions of the cosmological time. Its integration only occurs under additional assumptions.In a large number of known solutions the linear equation of state $p = (\gamma-1)\rho$ is adopted. Such linear relation reduces the system of field equations to a single non-linear second order differential equation for the scale factor.

The behavior of its solutions strongly depends on the pair (ϵ,γ) as the chronology of known solutions[3,13] illustrates: Einstein's[3] static solution (+1,2/3); De Sitter's[4] stationary solution (0,0); Friedmann's[5,7] evolutionary dust models (±1,1); etc.. Qualitative analysis of admissible models (taking into account the cosmological term) were done by Robertson[10] and Harrison[11]. Useful compendia of solutions are presented in Harrison[11] and Vajk[12]. Neglecting cosmological constant and making use of the conformal form of the metric, both of them determine parametric solution for arbitrary γ and $\varepsilon = \pm 1$. However, the cosmological time is left as unsolved integrals[11,12,14]. A quite distinct approach is that of Tauber[13], who makes use of the manifestly conformally flat form of the metric to construct solutions.

The specialized literature and textbooks on the subject[15] present the homogeneous and isotropic cosmologies by means of a representative set of particular solutions, i.e., those obtained when one a priori gives specific values for ε and γ ; It should be noticed the lack of an explicit solution including arbitrary values of both the parameters.

This work determines the general and unified solution for perfect fluid homogeneous and isotropic cosmologies with a linear equation of state: the non-linear differential equation for the scale factor $a(t)$ is transformed into a hypergeometric equation describing the cosmological time $t(a)$. The curvature parameter ε and the index γ appear in the general solution for $t(a)$ as continuous parameters. Therefore, all known solutions are specializations of the general solution through adequate choice of values for ε and γ. Negative values of pressure are needed to obtain those models originally developed with the aid of the cosmological term in the field equations. This is contradictory with the phenomenological prescription $1 \leq \gamma \leq 2$. However, it is consistent with the comprehensive theoretical considerations that take the energy conditions as requirements to be fulfilled by physical fluids[16]. Nevertheless, the general solution presented is valid whatever the supposed interval admissible to γ.

The next section is dedicated to the direct integration of the field equations for arbitrary ε and γ, which gives rise to a restricted solution. Section 3 exhibits the transformation of the nonlinear differential equation for the scale factor a(t) into a hypergeometric equation describing the inverse problem t(a) and the determination of all of its solutions. Those expressed by elementary functions are identified in section 4, which also describes a mechanism to generate solutions based on a property of the hypergeometric functions (complementary applications are supplied in an appendix). The complete solution obtained by making use of the conformal form of the metric is found in section 5.

2 THE RESTRICTED FORM OF THE UNIFIED SOLUTION

Spatially homogeneous and isotropic spacetimes are described by Robertson-Walker's line element, which may be taken in the following form:

$$ds^2 = dt^2 - a^2(t)[d\chi^2 + \sigma^2(\chi)(d\theta^2+\text{sen}^2\theta d\phi^2)] \ . \qquad (2.1)$$

The function $\sigma(\chi)$ is given by[15,a]

$$\sigma(\chi) = \frac{\sin\sqrt{\varepsilon}\,\chi}{\sqrt{\varepsilon}} \ , \qquad (2.2)$$

where ε is a parameter related to the constant curvature of the spatial sections. If positive, null or negative, ε is associated with closed, flat or open models[17], respectively, and it is usual to normalize its values to +1, 0 and -1.

However, from the strict point of view of the search for a unified solution, it is desirable to consider ε as an arbitrarily positive parameter which can be continuously deformed toward zero and negative values. The unified solution is only required to be well defined as ε moves and normalized values may be then taken for the sake of simplicity. The physical mechanism that would give rise to such a change in geometrical properties of space, if there is any, is another question not considered here.

Einstein's field equations for a perfect fluid with energy density ρ and isotropic pressure p are given by

$$\kappa\rho = \frac{3}{a^2}(\dot{a}^2+\varepsilon) \quad , \tag{2.3.a}$$

$$\kappa p = -2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \frac{\varepsilon}{a^2} \quad , \tag{2.3.b}$$

where $\kappa = 8\pi G$ and $c = 1$. A dot is used to indicate derivative with respect to the cosmological time t measured by a comoving observer.

Adding the linear equation of state

$$p = (\gamma-1)\rho \quad , \tag{2.4}$$

the system of field equations (2.3) is reduced to a single non-linear second-order differential equation for the scale factor a(t), given by

$$a\,\ddot{a} + \left(\frac{3\gamma-2}{2}\right)\dot{a}^2 + \varepsilon\left(\frac{3\gamma-2}{2}\right) = 0 \quad , \tag{2.5}$$

whose first integral is

$$(\dot{a})^2 = \left(\frac{a_0}{a}\right)^{3\gamma-2} - \varepsilon \quad , \tag{2.6}$$

where $c_\gamma = a_0^{3\gamma-2}$ is an integration constant with a_0 real and positive. This particular supposed form of the integration constant bears on some simple facts. It states not only that $c_\gamma = a_0^{3\gamma-2}$ is a specific integration constant for each matter content in the models, but also that the unified solution requires that all curves, describing the solutions for any pair (ε,γ), share the point $a = a_0$ at some instant of cosmological time. Furthermore, the derivative $\dot{a}(a_0) = \sqrt{1-\varepsilon}$ is the same for all γ and indicates the existence of an extremum for closed models. For any real and positive a_0 there is a maximum at $a = a_0$ if $\gamma > 2/3$, a minimum if $\gamma < 2/3$ and a stationary point if $\gamma = 2/3$ [cf. (2.5)]. Finally, the weak energy condition is fulfilled, since

$$\rho = \frac{3}{a_0^2}\left(\frac{a_0}{a}\right)^{3\gamma} > 0 \quad . \tag{2.7}$$

The integration of (2.6) is achieved with the introduction of an auxiliary variable defined by

$$u = \varepsilon\left(\frac{a}{a_0}\right)^{3\gamma-2} \quad . \tag{2.8}$$

Notice that the new variable is meaningless if $\varepsilon = 0$ and/or $\gamma = 2/3$. However, the corresponding solutions to these values of parameters are limiting cases of the general solutions and their detailed discussion is postponed until section 4. It should be remembered that a similar question arose in connection with eq. (2.2): the auxiliary variable $\sqrt{\varepsilon}\,\chi$ was meaningless for $\varepsilon = 0$. However, $\chi = \lim_{\varepsilon\to 0} \sigma(\chi)$ is a well defined function obtained as ε moves.

With the aid of transformation (2.8) the first integral (2.6) is written

$$\dot{u} = \frac{2}{(2A-1)a_0}\,\varepsilon^{A}\,u^{(1-A)}\,(1-u)^{1/2} \quad , \tag{2.9}$$

where

$$A = \frac{1}{2}\left(\frac{3\gamma}{3\gamma-2}\right) , \tag{2.10}$$

and it is readily integrable to furnish

$$t-t_0 = \left(\frac{2A-1}{2}\right) a_0\,\varepsilon^{-A}\int_{u_0}^{u} u^{(A-1)}\,(1-u)^{-1/2}\,du \quad , \tag{2.11}$$

which takes the form of an incomplete beta function through the choice $u(t_0) = u_0 = 0$. Such a function is related to Gauss hypergeometric function[18] and eq. (2.11) is expressed as:

$$t-t_0 = \left(\frac{2A-1}{2A}\right) a_0\;\varepsilon^{-A}\,u^{A}\,F(\tfrac{1}{2},A;A+1;u) \ , \tag{2.12}$$

which converges absolutely for $|u| \le 1$ and it is not defined if $A = 1-n$, $\gamma = \frac{4}{3}\left(\frac{1-n}{1-2n}\right)$, $n = 1,2,3\ldots$. It is named the restricted form of the unified solution[19], since it excludes an infinite family of solutions as a consequence of a specific choice of integration limits in the definite integral (2.11). Notice that the De Sitter's universes do not fulfill the initial condition adopted, being the first solutions excluded in the above sequence.

3 THE GENERAL FORM OF THE UNIFIED SOLUTION

A procedure in which the initial conditions remain arbitrary is demanded in order to get solutions for all γ . This is achieved when one considers the inverse problem, i.e., the solutions for a differential equation expressing the cosmological time as a function of the auxiliary variable u.

Inversion is made through (2.9). It provides the first derivative of $t(u)$,

$$\frac{dt}{du} = \left(\frac{2A-1}{2}\right) a_0 \, \varepsilon^{-A} \, u^{(A-1)} \, (1-u)^{-1/2} \quad , \tag{3.1}$$

whereas the second derivative may be put in the form

$$u(1-u) \, \frac{d^2t}{du^2} + \left[(1-A) - \left(\frac{3}{2} - A\right)u\right]\frac{dt}{du} = 0 \quad , \tag{3.2}$$

which is a hypergeometric equation having $a = 0$, $b = \frac{1}{2} - A$ and $c = 1-A$ as parameters[18,20]. If c is non integral the hypergeometric equation has the two linearly independent solutions $F(a,b;c;u)$ and $u^{(1-c)}F(a-c+1,b-c+1;2-c;u)$. However, the two solutions become identical if $c = 1$, and if $c \neq 1$ is any other integer, one of them becomes meaningless. In any of these cases the logarithmic solutions of the hypergeometric equation provide the other linearly independent solution.

Therefore, the general solution of (3.2) is given by (cf. ref. [20]):

$$t(u) = \alpha_0 + \beta_0 \, u^A \, F(\tfrac{1}{2},A;A+1;u) \qquad (1-A \neq n) \qquad (3.3.a)$$

$$t(u) = \sigma_{(n)} + \delta_{(n)} \left[- \sum_{\nu=1}^{n-1} \frac{\left(n-\frac{1}{2}\right)_{-\nu}}{\nu (n)_{-\nu}} u^{-\nu} + \sum_{\nu=1}^{\infty} \frac{\left(n-\frac{1}{2}\right)_{\nu}}{\nu (n)_{\nu}} u^{\nu} + \ln u \right] \qquad (1-A = n) \, , \qquad (3.3.b)$$

where $n = 1,2,3...$, $\sum_{\nu=1}^{n-1} = 0$ if $n = 1$, and $\alpha_0, \beta_0, \sigma_{(n)}$ and $\delta_{(n)}$ are constants.

Actually only one initial condition is arbitrary: the constants α_0 and $\sigma_{(n)}$ representing shifts in the origin of the time scale. The constants β_0 and $\delta_{(n)}$ are not arbitrary and are determined on the condition that (3.1) is the derivative of the solutions for t(u). Therefore:

a) Taking the derivative of (3.3.a) (cf. refs. [21] and [22]) one gets

$$\beta_0 = \left(\frac{2A-1}{2A}\right) a_0 \, \varepsilon^{-A} \quad . \qquad (3.4.a)$$

b) Using the relations

$$\frac{\left(n-\frac{1}{2}\right)_{\pm\nu}}{(n)_{\pm\nu}} = 2^{\mp 2\nu} \frac{[(n-1)!]^2 \; [2(n\pm\nu-1)]!}{[2(n-1)! \; [(n\pm\nu+1)]^2} \quad ,$$

obtained with the aid of Legendre's duplication formula for gamma functions[23], and

$$F(\tfrac{1}{2},b;b;u) = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2} u^k = (1-u)^{-1/2} \quad ,$$

where the summation indices are related by $\pm\nu = k-n+1$, one gets from the derivative of (3.3.b):

$$\delta_{(n)} = \frac{[2(n-1)]!}{[(n-1)!]^2}\left(\frac{1-2n}{2^{2n-1}}\right) a_0\ \varepsilon^{n-1} \quad . \tag{3.4.b}$$

The analytic continuations of (3.3.a,b) are obtained as follows (cf. ref. [20]):

a) Eq. (3.2) is transformed into a hypergeometric differential equation with parameters $a = 0$, $b = \frac{1}{2} - A$ and $c = 1/2$, if the substitution $u' = 1-u$ is made. It follows that (3.2) has the solution:

$$t(u) = \alpha_1+\beta_1(1-u)^{1/2}\ F\left(1-A,\ \tfrac{1}{2};\ \tfrac{3}{2};\ 1-u\right) \tag{3.5}$$

where α_1 is an arbitrary constant and

$$\beta_1 = (1-2A)\ a_0\ \varepsilon^{-A} \quad . \tag{3.6}$$

b) The substitution $u' = 1/u$ transforms (3.2) into a hypergeometric equation with parameters $a = 0$, $b = A$ and $c = A+\frac{1}{2}$. It follows that (3.2) has the solution:

$$t(u) = \alpha_2+\beta_2\, u^{(2A-1)/2}\ F\left(\tfrac{1}{2},\ \tfrac{1-2A}{2};\ \tfrac{3-2A}{2};\ \tfrac{1}{u}\right) \quad \left(A \neq n+\tfrac{1}{2}\right) \tag{3.7.a}$$

$$t(u) = \overline{\sigma}_{(n)} + \overline{\delta}_{(n)}\left[-\sum_{\nu=1}^{n}\frac{\left(n+\frac{1}{2}\right)_{-\nu}}{\nu(n+1)_{-\nu}}\left(\frac{1}{u}\right)^{-\nu} + \sum_{\nu=1}^{\infty}\frac{\left(n+\frac{1}{2}\right)_{\nu}}{\nu(n+1)_{\nu}}\left(\frac{1}{u}\right)^{\nu} + \ln\left(\frac{1}{u}\right)\right] \quad \left(A = n+\tfrac{1}{2}\right)\ , \tag{3.7.b}$$

where n = 1,2,3,..., α_2 and $\bar{\sigma}_{(n)}$ are arbitrary constants and β_2, $\bar{\delta}_{(n)}$ are determined analogously to (3.4.a,b).

Clearly the use of one of the solutions (3.3.a,b), (3.5) or (3.7.a,b), with a suitable adjustment of constants, is a matter of convenience: each one of them is a representation for the integral (2.11) up to an arbitrary additive constant, because their derivatives coincide.

All models may share a common time scale through an adequate choice for constants. A simple choice is t(0) = 0, but it is inconvenient for models with $\varepsilon > 0$ and $\gamma < 2/3$, for which a_0 is a minimum. However, since a_0 is a universal value for the scale factor in the unified approach, the time scale may be defined as $t(a_0) = t_0$. By this way, t_0 may be suitably defined for each ε and γ so as to recover the solutions presented in literature.

One should notice that the solution (3.5) has no restrictions upon the values of γ , whereas (3.3.a,b) and (3.7.a,b) have distinct forms according to the value of γ considered, and so it will be named the general form of the unified solution. Making the choice $t(a_0) = t_0$, (3.5) may be written:

$$t-t_0 = \frac{-2\ a_0}{(3\gamma-2)\,\varepsilon^A}\left\{\left[1-\varepsilon\left(\frac{a}{a_0}\right)^{3\gamma-2}\right]^{1/2} F\left(1-A,\ \frac{1}{2};\ \frac{3}{2};\ 1-\varepsilon\left(\frac{a}{a_0}\right)^{3\gamma-2}\right)\right.$$
$$\left. -(1-\varepsilon)^{1/2}\ F\left(1-A,\ \frac{1}{2};\ \frac{3}{2};\ 1-\varepsilon\right)\right\} . \qquad (3.8)$$

Notice, by the way, that the solutions strongly depend upon the values of the parameter A, whose behavior is shown in fig. 1. The continuity of A in each of its branches suggests that solutions characterized by slightly different values of γ

may be continuously deformed into one another. Notice, however, that A diverges at $\gamma = 2/3$ (the index γ for Einstein universe).

Figure 1

4 ELEMENTARY SOLUTIONS AND GENERATING TECHNIQUE

This section describes all the cases where the general and unified solution is reducible to elementary functions: not only those obtained by limiting processes (such as $\varepsilon \to 0$ or $\gamma \to 2/3$) but also all those for which the general and unified solution coincides with the power series expansion of some elementary function.

Consider the study of flat solutions, taken as assymptotic limits of the general solution when $\varepsilon \to 0$. One has the following expression, readily obtained from (3.8):

$$a(t) = a_0 \left[1 + \frac{3\gamma}{2} \left(\frac{t-t_0}{a_0}\right)\right]^{2/3\gamma} . \tag{4.1}$$

Making the choice $t_0 = 2a_0/3\gamma$ one recovers the restricted form of flat solutions presented in the specialized literature[11,14]. However, additional information is provided by (4.1): it has the exponential function[24] as limit when $\gamma \to 0$, and therefore also includes De Sitter's flat solution[25].

It should be noticed that ε and γ are independent parameters and two successive limiting processes commute. For instance, take $\gamma = 0$ ($A = 0$) in (3.8) (cf. ref. [26]), to get after some manipulation:

$$t-t_0 = a_0 \ln\left[\frac{\left(\frac{a}{a_0}\right) + \sqrt{\left(\frac{a}{a_0}\right)^2 - \varepsilon}}{1+\sqrt{1-\varepsilon}}\right] = \begin{cases} a_0 \operatorname{arccosh}\left(\frac{a}{a_0}\right) & \varepsilon = +1 \\ a_0 \ln\left(\frac{a}{a_0}\right) & \varepsilon = 0 \\ a_0\left[\operatorname{arcsinh}\left(\frac{a}{a_0}\right) - \operatorname{arcsinh} 1\right] & \varepsilon = -1 \end{cases} \qquad (4.2)$$

which unifies solutions of De Sitter type[4,6,10]. It is also illustrative of the requirements,made by the unified approach , of a common origin for the time scale, as pictured in fig. 2.

<u>Figure 2</u>

Reduction of the general unified solution (for $\varepsilon \neq 0$) to elementary functions is found with the aid of the integral (2.11). It is expressed by elementary functions only in the following cases[27]:

$$A = p \rightarrow \begin{cases} \gamma_n^{(S)} = \frac{4}{3}\left(\frac{1-n}{1-2n}\right) \\ \gamma_n^{(R)} = \frac{4}{3}\left(\frac{n}{2n-1}\right) \end{cases} \qquad \begin{matrix} p = 0, \pm 1, \pm 2, \ldots \\ n = 1, 2, 3, \ldots \end{matrix}$$

$$(4.3)$$

$$A = p + \frac{1}{2} \rightarrow \begin{cases} \gamma_n^{(D)} = \frac{1}{3}\left(\frac{1+2n}{n}\right) \\ \gamma_n^{(1/3)} = \frac{1}{3}\left(\frac{2n-1}{n}\right) \end{cases} \qquad \begin{matrix} p = 0, \pm 1, \pm 2, \ldots \\ n = 1, 2, 3, \ldots \end{matrix}$$

where the change in counting number was a matter of convenience. It is done in order to emphasize that each sequence starts from the index γ for an established elementary solution: De Sitter, radiation, dust and $\gamma = 1/3$, respectively indicated by a superscript. Notice that all of them satisfy $\lim_{n\to\infty} \gamma_n = \frac{2}{3}$ and therefore the solutions for the $\gamma_n^{(S)}$ and $\gamma_n^{(1/3)}$ sequences are consistent with the dominant energy condition ($|p| \leq \rho$), while those for the $\gamma_n^{(R)}$ and $\gamma_n^{(D)}$ sequences are consistent with the strong energy condition ($\rho + 3p \geq 0$). In any case, the weak energy condition ($\rho \geq 0$) is taken for granted whatever the value of γ [cf.(2.7)]. Solutions corresponding to values of γ apart from those indicated in (4.3) are given by a power series solution, i.e., a hypergeometric function, which may describe, or not, some special function.

Noticing that any two values of A differ in an integer number, the cosmological solutions corresponding to any consecutive values of γ in each sequence are represented by two contiguous hypergeometric functions, i.e., those whose corresponding parameters differ in a unity. Therefore, a generating technique for solutions may be devised, since between any two contiguous functions there exists a linear relation with coefficients which are linear functions of the argument[18].

A selective criterion is to pick up among the existing fifteen contiguity relations that one in which: a) the third parameter is decreased in a unity; b) the first (or second) parameter is increased (or decreased) in a unity if it decreases (increases) in a given sequence.

Application to one of the sequences makes the process self-evident. For the radiation sequence $\gamma_n^{(R)} = \frac{4}{3}\left(\frac{n}{2n-1}\right)$, $A = n$, and (3.5) is written:

$$t(u_{(n)}) = \alpha_1 + \beta_1 \left(1-u_{(n)}\right)^{1/2} F\left(1-n,\ \frac{1}{2};\ \frac{3}{2};\ 1-u_{(n)}\right) \quad , \qquad (4.4)$$

where $u_{(n)} = \varepsilon\left(\frac{a}{a_0}\right)^{2/(2n-1)}$. The contiguity relations furnish[28]:

$$(1-n)F\left(2-n,\frac{1}{2};\frac{3}{2};1-u_{(n)}\right) + \left(n-\frac{1}{2}\right)F\left(1-n,\frac{1}{2};\frac{3}{2};1-u_{(n)}\right) - \frac{1}{2}\,u_{(n)}^{n-1} = 0 \quad , \qquad (4.5)$$

and it follows that:

a) if n = 1, (4.5) is an identity,

b) if n = 2, $\frac{3}{2}\,F\left(-1,\frac{1}{2};\frac{3}{2};1-u_{(2)}\right) = \frac{1}{2}\,u_{(2)} + 1,$

c) if n = 3, $\frac{5}{2}\,F\left(-2,\frac{1}{2};\frac{3}{2};1-u_{(3)}\right) = \frac{1}{2}\,u^2_{(3)} + u_{(3)} + 2,$

and so on...

Therefore the successive elementary functional forms for the hypergeometric function in (4.4) are generated through (4.5), whereas the functional dependence of u(a) changes at each step. Specializing to get t(0) = 0, the first three solutions for the radiation sequence are:

a) $n = 1,\ \gamma = \frac{4}{3},\ p = \frac{1}{3}\rho$ (Tolman's radiation solutions[8])

$$t = \frac{a_0}{\varepsilon}\left\{1 - \left[1 - \varepsilon\left(\frac{a}{a_0}\right)^2\right]^{1/2}\right\} \quad ; \qquad (4.6)$$

b) $n = 2,\ \gamma = \frac{8}{9},\ p = -\frac{1}{9}\rho$

$$t = \frac{2a_0}{\varepsilon^2}\left\{1-\left[1-\varepsilon\left(\frac{a}{a_0}\right)^{2/3}\right]^{1/2} - \frac{\varepsilon}{2}\left(\frac{a}{a_0}\right)^{2/3}\left[1-\varepsilon\left(\frac{a}{a_0}\right)^{2/3}\right]^{1/2}\right\} \; ; \qquad (4.7)$$

c) $n = 3,\ \gamma = \frac{4}{5},\ p = -\frac{1}{5}\rho$

$$t = \frac{8a_0}{3\varepsilon^3}\left\{1-\left[1-\varepsilon\left(\frac{a}{a_0}\right)^{2/5}\right]^{1/2} - \frac{\varepsilon}{8}\left(\frac{a}{a_0}\right)^{2/5}\left[1-\varepsilon\left(\frac{a}{a_0}\right)^{2/5}\right]^{1/2} \cdot \left[4 + 3\varepsilon\left(\frac{a}{a_0}\right)^{2/5}\right]\right\} \quad . \qquad (4.8)$$

Other sequences of elementary solutions are examined in the appendix

Finally, consider the case $\gamma = 2/3$. The corresponding solutions for arbitrary ε are obtained from the general solution when $A \to \infty$. The simplest way to do it is from the solution (3.3.a), making use of

$$\lim_{A\to\infty} F(\tfrac{1}{2},A;A+1;u) = (1-\varepsilon)^{1/2} \quad .$$

Therefore, it follows from (3.3.a) that:

$$a = a_0 + \sqrt{1-\varepsilon}\,(t-t_0) \qquad \left(\gamma = \frac{2}{3}\right) \quad , \tag{4.9}$$

which is readily obtained from direct integration of (2.6). If $\varepsilon = 1$, (4.9) reproduces Einstein's static universe[3]. If $0 \leq \varepsilon < 1$, $\varepsilon = 0$ or $\varepsilon < 0$, it gives rise to a closed, flat or open, expansionist solution [11], respectively. For all these models $\ddot{a} = 0$ and the rate of expansion, $\dot{a} = \sqrt{1-\varepsilon}$, depends upon the value of ε, as a consequence of the unified approach.

5 The Conformal Form of the Solutions

Differently from flat solutions, for which $t(a)$ is readily invertible to furnish $a(t)$, one cannot in general invert the solutions with $\varepsilon \neq 0$.

Some gain in simplicity is achieved when one introduces the conformal time τ, defined by

$$dt = a(\tau)d\tau \quad , \tag{5.1}$$

through which the metric takes the conformal form

$$ds^2 = a^2(\tau)[d\tau^2 - d\chi^2 - \sigma^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)] \quad . \tag{5.2}$$

Under the conformal transformation the second-order differential equation (2.5) is translated to the form

$$aa'' + \left(\frac{3\gamma-4}{2}\right)a'^2 + \left(\frac{3\gamma-2}{2}\right)\varepsilon a^2 = 0 \quad , \tag{5.3}$$

and its first integral is expressed by

$$\left(\frac{a'}{a}\right)^2 = \left(\frac{a_0}{a}\right)^{3\gamma-2} - \varepsilon \quad , \tag{5.4}$$

where a prime denotes derivative with respect to τ .

Making the transformations[29]

$$z = \ln a, \quad \text{for} \quad \gamma = 2/3 \ , \quad \text{and} \tag{5.5}$$

$$z = a^{(3\gamma-2)/2} \quad , \ \gamma \neq 2/3 \quad , \tag{5.6}$$

eq. (5.3) is translated to the forms

$$z'' = 0 \quad , \qquad \text{for } \gamma = 2/3 \quad , \tag{5.7}$$

and

$$z'' + \varepsilon\left(\frac{3\gamma-2}{2}\right)^2 z = 0 \quad , \quad \text{for } \gamma \neq \frac{2}{3} \quad . \tag{5.8}$$

Therefore, if $\gamma = 2/3$,

$$a(\tau) = a_0 \, e^{\sqrt{1-\varepsilon}\,(\tau-\tau_0)} \quad ,$$

and the integration of (5.1) gives

$$t(\tau) - t_0 = \frac{a_0}{\sqrt{1-\varepsilon}}\left(e^{\sqrt{1-\varepsilon}\,(\tau-\tau_0)} - 1\right) \quad ,$$

both standing for the conformal form of the solution (4.9).

The interesting cases occur when $\gamma \neq 2/3$ for which, (5.3) is transformed into the equation describing the motion of a classical unitary mass particle subject to a linear force. If $\varepsilon > 0$ or $\varepsilon < 0$ the force is of restoring or repulsive type, respectively, while the motion of a free particle corresponds to $\varepsilon = 0$. The general solution of (5.8) is obviously:

$$z = \frac{z_0}{\sqrt{\varepsilon}} \sin \sqrt{\varepsilon}\left(\left|\frac{3\gamma-2}{2}\right|\tau + \delta\right) \quad , \tag{5.9}$$

where the oscillator (anti-oscillator) frequency is given by

$$\omega = \left|\frac{3\gamma-2}{2}\right| \quad , \tag{5.10}$$

and z_0, δ are integration constants.

The first integral (5.4) is translated, by means of the transformation (5.6), into the energy equation

$$\frac{1}{2}(z')^2 + \frac{1}{2}\varepsilon\omega^2 z^2 = \frac{1}{2}\omega^2 a_0^{3\gamma-2} \quad , \tag{5.11}$$

used to determine $z_0 = a_0^{(3\gamma-2)/2}$.

There is no loss of generality if one chooses $\delta = 0$. Thus the solution for the scale factor $a(\tau)$ is

$$a(\tau) = a_0 \left(\frac{\sin\sqrt{\varepsilon}\left|\frac{3\gamma-2}{2}\right|\tau}{\sqrt{\varepsilon}}\right)^{\frac{2}{3\gamma-2}} \quad , \tag{5.12}$$

and (5.1) furnishes the cosmological time $t(\tau)$:

$$t = a_0 \int \left(\frac{\sin\sqrt{\varepsilon}\omega\tau}{\sqrt{\varepsilon}}\right)^{\frac{2}{3\gamma-2}} d\tau + \text{constant} \; . \tag{5.13}$$

The last integral is left unsolved in literature without any references to the explicit form of its representations[11,12,14].

However, it is easy to verify that (5.13) is just the integral (2.11) with the auxiliary variable u defined by:

$$u = \varepsilon \left(\frac{a}{a_0}\right)^{3\gamma-2} = \sin^2\sqrt{\varepsilon}\ \omega\tau \ . \qquad (5.14)$$

Therefore, the solutions of the hypergeometric equation (3.3.a,b), (3.5) and (3.7.a,b), are representations for (5.13) merely by substitution of the relation (5.14). As an example,the conformal forms of the first three solution in the radiation series [cf. (4.6), (4.7) and (4.8)] are:

a) $n = 1,\ \gamma = \frac{4}{3},\ \ p = \frac{1}{3}\rho$ (Tolman's radiation solutions[8])

$$a(\tau) = a_0 \left(\frac{\sin\sqrt{\varepsilon}\ \tau}{\sqrt{\varepsilon}}\right) \ ,$$

$$t(\tau) = \frac{a_0}{\varepsilon}\ (1-\cos\sqrt{\varepsilon}\ \tau) \ ;$$

b) $n = 2,\ \gamma = \frac{8}{9},\ p = -\frac{1}{9}\rho$

$$a(\tau) = a_0 \left(\frac{\sin\sqrt{\varepsilon}\ \tau/3}{\sqrt{\varepsilon}}\right)^3 \ ,$$

$$t(\tau) = \frac{a_0}{\varepsilon^2}\left[2 - 3\cos\sqrt{\varepsilon}\ \frac{\tau}{3} + \cos^3\sqrt{\varepsilon}\ \ \frac{\tau}{3}\right] ;$$

c) $n = 3,\ \gamma = \frac{4}{5},\ p = -\frac{1}{5}\rho$

$$a(\tau) = a_0 \left(\frac{\mathrm{sen}\sqrt{\varepsilon}\ \ \tau/5}{\sqrt{\varepsilon}}\right)^5 \ ,$$

$$t(\tau) = \frac{a_0}{\varepsilon^3}\left[\frac{8}{3} - 5\cos\sqrt{\varepsilon}\ \frac{\tau}{5} + \frac{10}{3}\cos^3\sqrt{\varepsilon}\ \frac{\tau}{5} - \cos^5\sqrt{\varepsilon}\ \frac{\tau}{5}\right] .$$

The formal analogy between closed models and harmonic oscillators points to the cyclic character of closed models. Inversion of "direction" of motion occurs when $z = z_0$, at the instant of the conformal time given by:

$$T = \frac{\pi}{2\omega} = \frac{\pi}{|3\gamma-2|} \qquad (\gamma \neq \frac{2}{3}) \quad . \tag{5.15}$$

It should be noticed that all closed models exhibit a maximum when described through the auxiliary conformal scale factor $z(\tau)$. However, one can define the "half-period" of closed models as the time needed to reach the maximum value for the scale factor, restricting oneself to models with $\gamma > 2/3$, for which a maximum for $a(\tau)$ actually exists [cf. (5.6)].

Bearing this definition in mind, the information contained in (5.15) may be translated back to cosmological time. Noticing that $u(T) = 1$, the general solution (3.3.a) provides for the "half period", as described in the cosmological time coordinate:

$$T_m = \left(\frac{2A-1}{2}\right) a_0 \sqrt{\pi} \frac{\Gamma(A)}{\Gamma(A+\frac{1}{2})} \qquad (\gamma > \frac{2}{3}) \quad , \tag{5.16}$$

where Γ stands for the gamma function[30].

An exact formula to calculate the life time of closed models is provided by eq. (5.16), whose plot is shown in Fig. 3. When the strong energy condition ($\rho+3p \geq 0$) replaces the more restrictive condition of positivity of pressure and energy, the life time of the models may increase indefinitely in the range of negative pressures[31].

Figure 3

Another interesting feature is that T_m decreases with increasing γ, i.e., the life time of closed models diminishes when the effective gravitational mass increases, as one should expect. A simple interpretation is available with the use of eq. (5.8). Suppose a set of spring — mass systems, with different spring constants, equally elongated of z_0. The spring constant $k = \left(\frac{3\gamma-2}{2}\right)^2$ increases with the increasing of the relation p/ρ, shortening the time needed to complete a half-cycle of motion.

6 CONCLUSIONS AND FINAL REMARKS

This work exhibits the general and unified solution to perfect fluid homogeneous and isotropic cosmological models in which the fluid obeys the equation of state $p = (\gamma-1)\rho$. The evolution of all models is described by the solutions of a hypergeometric equation. Considering ε and γ as continuous parameters, all information about the underlying geometry and matter content is conveyed into the general solution. This enables the comparison of the evolution of models constructed by the use of two slightly different values of γ, since the general solution furnishes the second solution as a neighbor curve whose separation from the first may be determined with accuracy.

The solutions correspond to power series expansions of elementary function when the values of γ are described by a set of four infinite numerable sequences. All these solutions are consistent with the energy conditions and recurrence relations among contiguous hypergeometric functions provide the generating mechanism to display all solutions in each sequence. Apart from these sequences, the solutions are described by hypergeometric functions, i.e., power series solutions (representing or not some other special function).

The power series feature makes the solutions useful for the analysis of the behavior near singularity. Considering the first two terms in solution (3.3.a) one has for $\gamma > 2/3$:

$$t \approx \frac{2a_0}{3\gamma} \left(\frac{a}{a_0}\right)^{3\gamma/2} \left[1 + \frac{\varepsilon A}{2(A+1)} \left(\frac{a}{a_0}\right)^{3\gamma-2}\right] ,$$

where the origin of time scale was choosen on the condition $t(0) = 0$. It should be noticed that the information carried by the first order term in the general solution is lost, when one approaches the singularity, by the use of the first integral eq. (2.6), as it is usually done [15c,16].

It should be remarked that one may give to analytical expressions for observational parameters of astrophysical interest the same unified treatment in the sense here presented.

By the use of conformal time the evolution of the models may be set in the form analogous to that of a classical particle submitted to a linear force. The behavior of perfect fluid homogeneous and isotropic cosmologies for any values of the index γ is analogous to that of a global oscillator if $\varepsilon = +1$, an anti-oscillator if $\varepsilon = -1$ and a free particle if $\varepsilon = 0$.

The scale factor for all models with $\gamma > 2/3$ exhibits a maximum, and the time to reach maximum radius decreases monotonically with increasing γ. Qualitative features of closed models behavior remain without change if the upper bound $\gamma \leq 2$, determined by local thermodynamic considerations, is surpassed.

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APPENDIX

This appendix applies the generating mechanism devised in section (4) to the sequences $\gamma_n^{(D)}$, $\gamma_n^{(S)}$ and $\gamma_n^{(1/3)}$.

I) $\gamma_n^{(D)} = \frac{1}{3}\left(\frac{1+2n}{n}\right)$, $A = n + \frac{1}{2}$, $n = 1,2,3,\ldots$

From solution (3.3.a) one has:

$$t = \alpha_0 + \beta_0 \, u_{(n)}^{n+\frac{1}{2}} \, F\left(\frac{1}{2},\ n+\frac{1}{2};\ n+\frac{3}{2};\ u_{(n)}\right) , \qquad (A.1)$$

where $u_{(n)} = \varepsilon\left(\frac{a}{a_0}\right)^{1/n}$. The contiguity relations[32] furnish:

$$F\left(\frac{1}{2},\frac{1}{2}+n;\frac{3}{2}+n;u_{(n)}\right) = \frac{2n+1}{2n}\,\frac{1}{u_{(n)}}\left[F\left(\frac{1}{2},n-\frac{1}{2};n+\frac{1}{2};u_{(n)}\right) - (1-u_{(n)})^{1/2}\right] . \qquad (A.2)$$

It follows that

a) if $n = 1$ one has[33]

$$F\left(\frac{1}{2},\frac{3}{2};\frac{5}{2};u_{(1)}\right) = \frac{3}{2u_{(1)}}\left[\frac{\sin^{-1}\sqrt{u_{(1)}}}{\sqrt{u_{(1)}}} - (1-u_{(1)})^{1/2}\right] ;$$

b) if $n = 2$

$$F\left(\frac{1}{2},\frac{5}{2};\frac{7}{2};u_{(2)}\right) = \frac{5}{4u_{(2)}}\left[F\left(\frac{1}{2},\frac{3}{2};\frac{5}{2};u_{(2)}\right)-(1-u_{(2)})^{1/2}\right].$$

Determining α_0 on the condition $t(0) = 0$, the first two solutions in this sequence are described by

a) $n = 1$, $\gamma = 1$, $p = 0$ (Friedmann's dust models[5,7,9])

$$t = \frac{a_0}{\varepsilon}\left[\frac{\mathrm{S}_{in}^{-1}\sqrt{\varepsilon}(a/a_0)^{1/2}}{\sqrt{\varepsilon}} - \left(\frac{a}{a_0}\right)^{1/2}\left(1 - \frac{a}{a_0}\right)^{1/2}\right] \quad , \text{(A.3)}$$

whose conformal form is

$$\begin{aligned} a(\tau) &= \frac{a_0}{\varepsilon}\sin^2\frac{\sqrt{\varepsilon}\tau}{2} = \frac{a_0}{2\varepsilon}(1-\cos\sqrt{\varepsilon}\,\tau) \quad , \\ t(\tau) &= \frac{2a_0}{\varepsilon}\left(\tau - \frac{\sin\sqrt{\varepsilon}\,\tau}{\varepsilon}\right) \quad , \end{aligned} \qquad \text{(A.4)}$$

b) $n = 2$, $\gamma = 5/6$, $p = -\frac{1}{6}\rho$

$$t = \frac{a_0}{\varepsilon^2}\left\{\frac{3}{2}\frac{\mathrm{S}_{in}^{-1}\sqrt{\varepsilon}(a/a_0)^{1/4}}{\sqrt{\varepsilon}} - \varepsilon\left(\frac{a}{a_0}\right)^{1/4}\left[\left(\frac{a}{a_0}\right)^{1/2} + \frac{3}{2\varepsilon}\right] \cdot \left[1 - \varepsilon\left(\frac{a}{a_0}\right)^{1/2}\right]^{1/2}\right\} \quad , \qquad \text{(A.5)}$$

given in the conformal form by

$$\begin{aligned} a(\tau) &= \frac{a_0}{\varepsilon^2}\sin^4\frac{\sqrt{\varepsilon}\tau}{4} \quad , \\ t(\tau) &= \frac{a_0}{\varepsilon^2}\left(\frac{3\tau}{8} - \frac{5}{4\sqrt{\varepsilon}}\sin\frac{\sqrt{\varepsilon}\tau}{2} + \frac{1}{\sqrt{\varepsilon}}\sin\frac{\sqrt{\varepsilon}\tau}{4}\cos^3\frac{\sqrt{\varepsilon}\tau}{4}\right). \end{aligned} \qquad \text{(A.6)}$$

II) $\gamma_n^{(S)} = \frac{4}{3}\left(\frac{1-n}{1-2n}\right)$, $A = 1-n$, $n = 1,2,3,\ldots$

From solution (3.5) one has

$$t = \alpha_1 + \beta_1(1-u_{(n)})^{1/2}F\left(n,\frac{1}{2};\frac{3}{2};1-u_{(n)}\right) \quad , \qquad \text{(A.7)}$$

where $u_{(n)} = \varepsilon \left(\frac{a_0}{a}\right)^{2/(2n-1)}$. The contiguity relations furnish[34]

$$(1-n)F\left(n,\tfrac{1}{2};\tfrac{3}{2};1-u_{(n)}\right) = \left(\tfrac{3}{2}-n\right)F\left(n-1,\tfrac{1}{2};\tfrac{3}{2};1-u_{(n)}\right) - \tfrac{1}{2}\,u_{(n)}^{n-1}\,, \tag{A.8}$$

and it follows that

a) if $n = 1$, the above relation is an identity,

b) if $n = 2$, one has[26]

$$F\left(2,\tfrac{1}{2};\tfrac{3}{2};1-u_{(2)}\right) = \frac{1}{2}\left[\frac{1}{u_{(2)}} + \frac{1}{2\sqrt{1-u_{(2)}}}\ln\frac{1+\sqrt{1-u_{(2)}}}{1-\sqrt{1-u_{(2)}}}\right].$$

The first two solutions are

a) $n = 1$, $\gamma = 0$, $p = -\rho$; the unified De Sitter's solutions given by (4.2);

b) $n = 2$, $\gamma = 4/9$, $p = -\frac{5}{9}\rho$

$$t-t_0 = \frac{3}{2}a_0\left[\varepsilon\ln\frac{1+\sqrt{1-\varepsilon(a_0/a)^{2/3}}}{1-\sqrt{1-\varepsilon(a_0/a)^{2/3}}} + \left(\frac{a}{a_0}\right)^{2/3}\sqrt{1-\varepsilon(a_0/a)^{2/3}} - \sqrt{1-\varepsilon}\right]. \tag{A.9}$$

III) $\gamma_n^{(1/3)} = \frac{1}{3}\left(\frac{2n-1}{n}\right)$, $A = \frac{1}{2}-n$, $n = 1,2,3,\ldots$

From (3.5) one has

$$t = \alpha_1+\beta_1(1-u_{(n)})^{1/2}F\left(\tfrac{1}{2}+n,\tfrac{1}{2};\tfrac{3}{2};1-u_{(n)}\right)\,, \tag{A.10}$$

where $u_{(n)} = \varepsilon\left(\frac{a_0}{a}\right)^{1/n}$. The contiguity relations provide[34]

$$\left(\frac{1}{2}-n\right)F\left(\frac{1}{2}+n,\frac{1}{2};\frac{3}{2};1-u_{(n)}\right) = (1-n)F\left(n-\frac{1}{2},\frac{1}{2};\frac{3}{2};1-u_{(n)}\right) - \frac{1}{2}\,u_{(n)}^{\frac{1}{2}-n} , \tag{A.11}$$

and it follows that:

a) if $n = 1$, the above relation reduces to an identity ;

b) if $n = 2$, $F\left(\frac{5}{2},\frac{1}{2};\frac{3}{2};1-u_{(2)}\right) = \frac{2}{3}\left[u_{(2)}^{-1/2} + \frac{1}{2}\,u_{(2)}^{-3/2}\right]$.

On the condition $t(a_0) = 0$, the first two solutions are

a) $n = 1$, $\gamma = \frac{1}{3}$, $p = -\frac{2}{3}\rho$

$$t = 2a_0\left[\sqrt{\frac{a}{a_0}-\varepsilon} - \sqrt{1-\varepsilon}\right] ; \tag{A.12}$$

b) $n = 2$, $\gamma = \frac{1}{2}$, $p = -\frac{1}{2}\rho$

$$t = \frac{8}{3}\,a_0\left\{\left[1-\varepsilon\left(\frac{a_0}{a}\right)^{1/2}\right]^{1/2}\left[\frac{1}{2}\left(\frac{a}{a_0}\right)^{3/4} + \varepsilon\left(\frac{a}{a_0}\right)^{1/4}\right] - \sqrt{1-\varepsilon}\left(\frac{1}{2}+\varepsilon\right)\right\} . \tag{A.13}$$

CAPTIONS FOR FIGURES

Fig. 1: Graphic representation of $A(\gamma) = \frac{1}{2}\left(\frac{3}{3\gamma-2}\right)$.

Fig. 2: Graphic representation of De Sitter's solutions when $t(a_0) = t_0 = 0$. The dotted line shows the open solution when the time scale is required to satisfy $t(0) = 0$, as it is usually done in literature.

Fig. 3: Graphic representation of the life time $2T_m(\gamma)$.

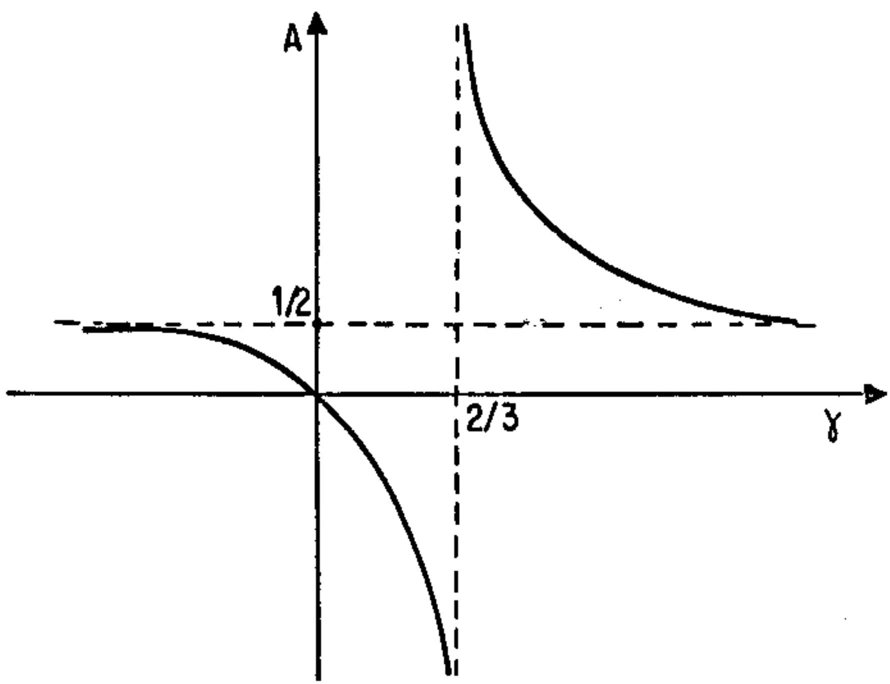

FIG.1

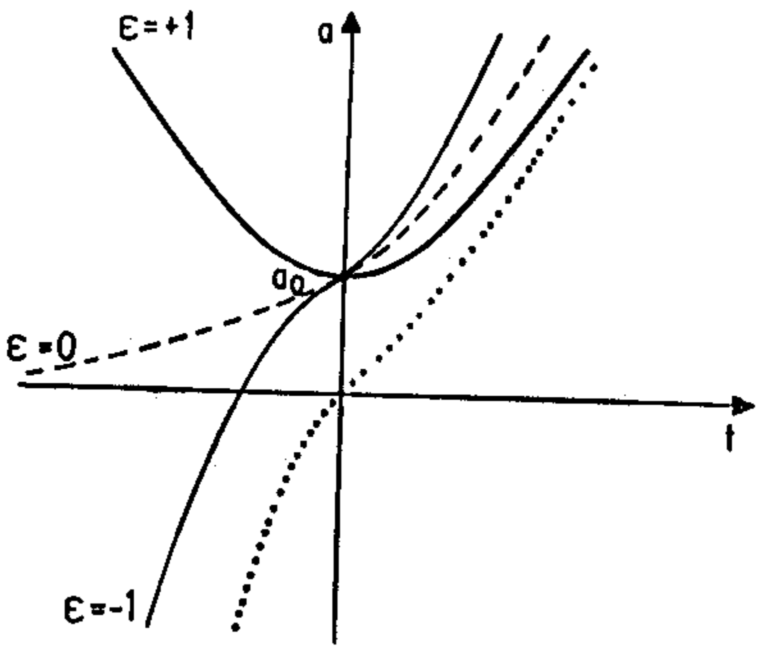

FIG.2

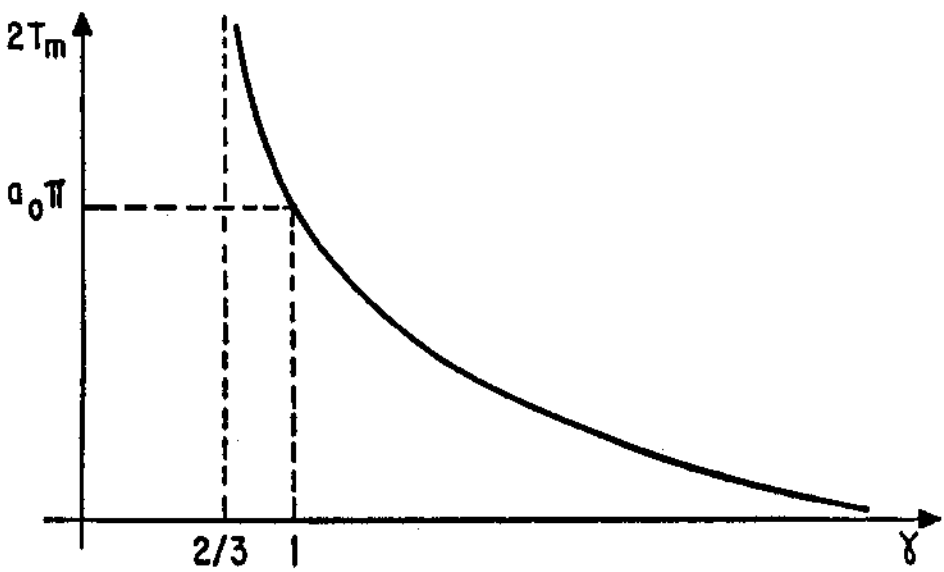

FIG.3

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