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SUPERSYMMETRIC KLEIN-GORDON EQUATION IN d-DIMENSIONS

by

C.G. Bollini and J.J. Giambiagi+

Centro Brasileiro de Pesquisas Físicas - CNPq/CBPF
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

+On leave at CERN, Genève.

ABSTRACT

The simple supersymmetric Wess-Zumino model is extended to arbitrary $d=4\nu$ dimensions. The components of the quiral superfield are found to obey a higher order Klein-Gordon-type equation, whose Green functions are discussed.

Key-words: Supersymmetric Klein-Gordon; Arbitrary dimensions.

1 INTRODUCTION

In general, when extending field theory to higher dimensions, the kinetic part of the evolution equation is kept as the usual second order Klein-Gordon equation. However this procedure is not unique. Other prescriptions are possible, in particular one which emerges naturally with the use of the supersymmetry algebra. In reference [1] this alternative was rised, but soon abandoned in favour of the common one. Instead, we will adopt here the alternative (called I in Ref. [1]) which corresponds to higher order equation in higher dimensions.

There are some good reason for this kind of extension, especially when generalizing dimensional regularization to supersymmetric theories. Not adopting this procedure might lead to important simplifications [2] but also to some inconsistencies [3].

2 THE CHIRAL MODEL

It is with those ideas in mind that we shall take the simple Wess-Zumino model [4] in higher dimensions. To avoid unnecessary technicalities we shall take the number of dimensions d as:

$$d = 4\nu \tag{1}$$

The number of components ω, of a Weyl-spinor [5] is

$$\omega = 2^{\frac{d}{2}-1} = 2^{2\nu-1} \tag{2}$$

The generators of simple supersymmetry obey the usual commutation relations. To these generators there correspond ω Grassman variables θ^{α} and their ω conjugate $\bar{\theta}^{\dot{\alpha}}$. Just as in four dimensions we can define superfields and represent the generators as derivative operators acting on them. Also, we can define the usual covariant derivatives[6], D_{α} and $\bar{D}_{\dot{\alpha}}$, leading to the definition of "chiral fields" as the solutions of

$$\bar{D}_{\dot{\alpha}}\phi = 0 \tag{3}$$

From which we obtain:

$$\Phi = e^{\frac{i}{2}\theta^{\alpha}\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\alpha}}\Phi_0(x,\theta) \tag{4}$$

where

$$\Phi_0 = \sum_{s=0}^{\omega}\frac{1}{s!}\theta^{\alpha_1}\cdots\theta^{\alpha_s}\psi_{\alpha_1\ldots\alpha_s}(x) \tag{5}$$

The $\psi_{\alpha_1\ldots\alpha_s}$ $(s=0,1\ldots\omega)$ are the $\omega+1$ components of the quiral superfield. They are antisymmetric, and we write, for $s=0$ and $s=\omega$:

$$\psi(x) = A(x) \tag{6}$$

$$\psi_{\alpha_1\ldots\alpha_\omega}(x) = \varepsilon_{\alpha_1\ldots\alpha_\omega}F(x) \tag{7}$$

where $\varepsilon_{\alpha_1\ldots\alpha_\omega}$ is the ω-dimensional Levi-Civita symbol.

3 THE LAGRANGIAN

As in four dimensions the variation of the highest component under a supersymmetry transformation is a divergence. So, the lagrangian for a chiral superfield can be written in the usual way

$$\mathcal{L} = \bar{\Phi}\Phi\Big|_D + c\,\Phi^2\Big|_F + \text{h.c.} + \text{interaction terms} \tag{8}$$

As the mass dimension of $\mathcal{L}$ is d ([m]=1, [$\mathcal{L}$]=d), from the first term in (8) we deduce:

$$[\Phi] = \frac{d-\omega}{2} \tag{9}$$

And, from the second one

$$[c] = \frac{\omega}{2} \tag{10}$$

So we write

$$c = \frac{1}{2}\, m^{\frac{\omega}{2}} \tag{11}$$

Now we use (4) and (5) to write (8) explicitly in component form. As a first step we write

$$\mathcal{L} = \bar{\Phi}_0\Phi_0\Big|_D + \bar{\Phi}_0\, i\theta^{\alpha}\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\Phi_0\Big|_D + \bar{\Phi}_0\,\frac{1}{2}\,(i\theta^{\alpha}\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}})^2\Phi_0\Big|_D + \ldots$$

$$\ldots + \frac{1}{2}\, m^{\frac{\omega}{2}}\,\Phi_0^2\Big|_F + \frac{1}{2}\, m^{\frac{\omega}{2}}\,\bar{\Phi}_0^2\Big|_F + \text{interaction} \tag{12}$$

Note that

$$\theta^{\alpha_1}\ldots\theta^{\alpha_\omega} = \varepsilon^{\alpha_1\ldots\alpha_\omega}\,\overset{\omega}{\theta} \tag{13}$$

with

$$\overset{\omega}{\theta} = \frac{1}{\omega!}\, \varepsilon_{\alpha_1\ldots\alpha_\omega} \theta^{\alpha_1}\cdots\theta^{\alpha_\omega} = \theta^1\ldots\theta^\omega \,. \qquad (14)$$

Then, the explicit separation of the "D" and "F" terms, gives

$$\mathcal{L} = \sum_{s=0}^{\omega} \frac{\varepsilon^{\alpha_1\ldots\alpha_\omega}\varepsilon^{\dot\alpha_1\ldots\dot\alpha_\omega}}{s!(\omega-s)!}\, \bar\psi_{\dot\alpha_1\ldots\dot\alpha_s}\, i\,\partial_{\alpha_{s+1}\dot\alpha_{s+1}}\ldots i\,\partial_{\alpha_\omega\dot\alpha_\omega}\psi_{\alpha_1\ldots\alpha_s} + \frac{m^{\frac{\omega}{2}}}{2} \sum_{s=0}^{\omega} \frac{1}{s!(\omega-s)!}\left(\varepsilon^{\alpha_1\ldots\alpha_\omega}\psi_{\alpha_1\ldots\alpha_s}\psi_{\alpha_{s+1}\ldots\alpha_\omega} + \text{h.c.}\right) + \text{int.} \qquad (15)$$

4 EQUATIONS OF MOTION

From (15) we deduce for the free case:

$$\frac{\varepsilon^{\alpha_1\ldots\alpha_\omega}}{s!}\, i\,\partial_{\alpha_{s+1}\dot\alpha_{s+1}}\ldots\psi_{\alpha_1\ldots\alpha_s} + m^{\frac{\omega}{2}}\,\psi_{\dot\alpha_{s+1}\ldots\dot\alpha_\omega} = 0 \quad (s=0,1,\ldots,\omega) \qquad (16)$$

In particular, for $s=0$ and $s=\omega$, eq.(16) gives:

$$\Box^{\frac{\omega}{2}} A + m^{\frac{\omega}{2}} F^* = 0 \qquad (17)$$

$$F + m^{\frac{\omega}{2}} A^* = 0 \qquad (18)$$

respectively. Where we have used (6), (7), and

$$\varepsilon^{\alpha_1\ldots\alpha_\omega}\, i\,\partial_{\alpha_1\dot\alpha_1}\ldots i\,\partial_{\alpha_\omega\dot\alpha_\omega} = \mathrm{Det}(i\partial_{\alpha\dot\alpha})\,\varepsilon_{\dot\alpha_1\ldots\dot\alpha_\omega} = \Box^{\frac{\omega}{2}}\,\varepsilon_{\dot\alpha_1\ldots\dot\alpha_\omega} \qquad (19)$$

Now, to find the equations of motion proceed in the following way: first multiply (16) by $\frac{\varepsilon^{\dot{\alpha}_1\ldots\dot{\alpha}_\omega}}{(\omega-s)!}\, i\partial_{\beta_1\dot{\alpha}_1}\ldots i\partial_{\beta_s\dot{\alpha}_s}$. Then take the adjoint of (16) eliminate the barred component, and finally use (19) to find, for any s:

$$(\square^{\frac{\omega}{2}} - m^\omega)\ \psi_{\alpha_1\ldots\alpha_s} = 0 \qquad (20)$$

Only for d=4 (ω=2) does (20) coincide with the Klein-Gordon equation.

In general, expression (20) means that the free propagators should contain the factor: $(p=(p^2)^{\frac{1}{2}})$

$$P = \frac{1}{p^\omega - m^\omega} \qquad (21)$$

5 THE GAUGE FIELD

The real gauge superfield V can be treated in an analogous way. In d=4 and for the abelian case we have (See reference[6])

$$\mathcal{L} = \varepsilon^{\alpha\beta}\, V D_\alpha \bar{D}^{2} D_\beta V\Big|_D \qquad (22)$$

Where, as in (13) and (14) we use the notation:

$$\overset{\omega}{D} = D^1\ldots D^\omega = \frac{1}{\omega!}\,\varepsilon_{\alpha_1\ldots\alpha_\omega} D^{\alpha_1}\ldots D^{\alpha_\omega} \qquad (23)$$

A natural supersymmetric generalization of (22) is [7]:

$$\mathcal{L}_V = \varepsilon^{\alpha_1\ldots\alpha_\omega}\, V D_{\alpha_1}\ldots D_{\alpha_{\frac{\omega}{2}}}\, \overset{\omega}{\bar{D}}\, D_{\alpha_{\frac{\omega}{2}+1}}\ldots D_{\alpha_\omega} V\Big|_D \qquad (24)$$

Now, using the identity[8]:

$$\varepsilon^{\alpha_1 \ldots \alpha_\omega} \sum_{s=0}^{\omega} \frac{(-1)^s}{s!(\omega-s)!} D_{\alpha_1} \ldots D_{\alpha_s} \bar{D}^{\omega} D_{\alpha_{s+1}} \ldots D_{\alpha_\omega} \equiv \Box^{\frac{\omega}{2}} \qquad (25)$$

and choosing the gauge in which

$$\varepsilon^{\alpha_1 \ldots \alpha_\omega} D_{\alpha_1} \ldots D_{\alpha_s} \bar{D}^{\omega} D_{\alpha_{s+1}} \ldots D_{\alpha_\omega} V = 0 \quad (\text{for } s \neq \tfrac{\omega}{2}) \qquad (26)$$

we arrive at:

$$\mathcal{L}_v = V \Box^{\frac{\omega}{2}} V \Big|_D \qquad (27)$$

Which means that the free equation of motion for the abelian gauge superfield V, is:

$$\Box^{\frac{\omega}{2}} V = 0 \qquad (28)$$

Leading to the propagator:

$$P_0 = \frac{1}{p^{\omega}} \qquad (29)$$

6 THE GREEN FUNCTIONS

We then see that our straightforward procedure gives, in a natural way, a generalization of the Klein-Gordon equation to higher dimensions which is of order ω in the space-time derivatives. So, it seems appropriate and convenient to gain some confidence with equations (20) and (28), the respective propagators (21) and (29), their Green functions, etc.

The fundamental solution of (28) is easily found by taking the Fourier transform of (29), giving[9]:

$$F(P_0) = \frac{2^{4\nu-\omega}\pi^{2\nu}}{i\Gamma(\frac{\omega}{2})}\,\Gamma(2\nu-\frac{\omega}{2})\,R^{\omega-4\nu} \tag{30}$$

Where

$$R^2 = \sum_{i=1}^{d-1} x_i^2 - x_0^2 \tag{31}$$

The behaviour of (30) as a function of R depends on the number of dimensions through the exponent

$$\alpha = \omega-4\nu = 2^{2\nu-1}-4\nu = 2^{\frac{d}{2}-1} - d \tag{32}$$

α is negative for d=4 (ν=1). It is zero for ν=2, and it is positive for any other integer value of ν greater than 2.

When taking the finite part of (30) for $\nu > 1$ (note the pole in $\Gamma(-\frac{\alpha}{2})$), we get on additional $\ln R$ factor[10], so we have

$$\nu = 1 \quad , \quad d = 4 \qquad\qquad G \sim \frac{1}{R^2} \tag{33}$$

$$\nu = 2 \quad , \quad d = 8 \qquad\qquad G \sim \ln R \tag{34}$$

$$\nu > 2 \quad , \quad d = 4\nu \qquad\qquad G \sim R^{\alpha}\ln R \quad (\alpha>0) \tag{35}$$

For $\nu > 1$ the Green function has a confining form; it grows with R.

In the massive case, we can write the propagator (21) as a series of massless ones:

$$P = \frac{1}{p^{\omega}-m^{\omega}} = \frac{1}{m^{\omega}} \sum_{\ell=1}^{\infty} (\frac{m}{p})^{\omega\ell} \tag{36}$$

Taking now the Fourier transform of (36) (cf. eq.(30)), and eliminating the pole part of each term, we arrive at the expression:

$$F(P) = \frac{(4\pi)^{2\nu}}{im^{\omega}R^{4\nu}} \sum_{\ell=1}^{\infty} \frac{(\frac{mR}{2})^{\omega\ell}}{\Gamma(\frac{\omega\ell}{2})\Gamma(\frac{\omega\ell}{2}+1-2\nu)} \{2\ell n\frac{mR}{2} - \psi(\frac{\omega\ell}{2}) - \psi(\frac{\omega\ell}{2}+1-2\nu)\} \tag{37}$$

To check that (37) is the Green function of (20), we apply the operator $\Box^{\frac{\omega}{2}}$. The first term (ℓ=1) is the fundamental solution of (28) [10], so that the result of operating with $\Box^{\frac{\omega}{2}}$ is just a δ-function. For the rest of the series we use:

$$\Box^{\frac{\omega}{2}} R^{\frac{\omega}{2}+\alpha} = 2^{\omega} \frac{\Gamma(\frac{\omega+\alpha+2}{2})\Gamma(\frac{\omega+\alpha+4\nu}{2})}{\Gamma(\frac{\alpha+2}{2})\Gamma(\frac{\alpha+4\nu}{2})} R^{\alpha} \tag{38}$$

and the equation obtained by taking the derivative of (38) with respect to α. In this way we can show that (37) is indeed the fundamental solution of (20).

In four dimensions (ν=1,ω=2), the first term of (37) should be separated and written as if $\frac{-\psi(o)}{\Gamma(o)} = 1$. Eq. (37) gives then the usual Green function for the Klein-Gordon equation.

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