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REMARKS ON THE TRIVIALITY PHENOMENA IN $\lambda\phi^4$-O(N)-FIELD THEORY

by

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ABSTRACT

We formulate the $\lambda\phi^4$-0(N) - field theory in random path space. By making use of this random path formulation we offer a probabilistic-topological argument for the triviality phenomena of the quoted theory for $D > 4$.

Key-words: Triviality phenomena; $\lambda\phi^4$-0(N)-field theory; Random path space.

In the last years the analysis of the infinite's renormalization in quantum field theories have received a renewed attention due mainly to Aizeman[1], Fröhlich[2] and Aragão et al.[3] rigorous work on the triviality phenomena of the (apparent non-renormalizable) $\lambda\phi^4$-0(N)-field theory for a space-time with dimensionality greater than four. All these studies are based in a common idea: the lattice version of the Symanzik's path reformulation for the model[4].

Our aim in this brief report is to present a path formulation of $\lambda\phi^4$-0(N) (symmetric phase) suitable for the analysis of the triviality phenomena. In addition we make some remarks on the $N\to 0$ "Polymer limit"[5,6].

Let us start our study by considering the (bare) generating functional of the Green's functions of the 0(N) (symmetric phase) $\lambda\phi^4$ field theory in a D-dimensional Euclidean space-time

$$Z[J^a(x)] = \int \prod_{a=1}^{N} d\mu[\Phi^a(x)].\exp\{-\frac{\lambda_0}{4}\int d^Dx(\sum_{a=1}^{N}\Phi^a(x)^2)^2(x) - \int d^Dx(\sum_{a=1}^{N} J^a(x)\,\Phi^a(x))\} \tag{1}$$

where $\Phi^a(x)$ denotes a N-component real scalar 0(N) field, (μ_0,λ_0) the (bare) mass and coupling parameters and the Gaussian functional measure in (1) is

$$\prod_{a=1}^{N} d\mu[\Phi^a(x)] = \prod_{a=1}^{N}\left[(\prod_{x\in R^D} d\Phi^a(x))\exp\{-\frac{1}{2}\int d^Dx(\sum_{a=1}^{N}(\partial_\mu\phi^a)^2(x) + \mu_0^2\sum_{a=1}^{N}(\Phi^a)^2(x)\}\right] \tag{2}$$

Now, in order to get an effective expression for the functional integrand in (2), where we can evaluate the Φ^a functional integrations, we write the interaction $\lambda\phi^4$ term in the following form

$$\exp\{-\frac{\lambda_0}{4}\int d^Dx\,(\sum_{a=1}^{N}(\Phi^a(x))^2)^2\} = \int d\mu[\sigma].\exp\{-i\int d^Dx\,\sigma(x)\,(\sum_{a=1}^{N}(\Phi^a(x))^2)\} \tag{3}$$

where $\sigma(x)$ is an auxiliary scalar field and the σ functional measure in (3) is given by

$$d\mu[\sigma] = (\prod_{x\subset R^D} d\sigma(x)\exp\{-\frac{1}{2}\int d^Dx\,\frac{2}{\lambda_0}\,\sigma^2(x)\} \tag{4}$$

with covariance

$$<\sigma(x_1)\sigma(x_2)>_\sigma = \int d\mu[\sigma]\sigma(x_1)\sigma(x_2) = \frac{\lambda_0}{2}\,\delta^{(D)}(x_1-x_2) \tag{5}$$

The last result allows us to consider the $\delta(x)$ field as a random gaussian potential with noise's strenght $\frac{\lambda_0}{2}$.[7]

After substitution of (4) into (2), we can evaluate explicitly the Φ-functional integrations since they are of gaussian type. We, thus, get the result

$$Z[J^a(x)] = \int d\mu[\sigma]\mathrm{Det}^{-N/2}(-\Delta+\mu_0^2-2i\sigma)$$
$$\exp\{\frac{1}{2}\int d^Dx d^Dy\;J^a(x)\,(-\Delta+\mu_0^2-2i\sigma)\,\delta_{ab}J^B(y)\} \tag{6}$$

At this point of our study we implement the main idea: by following Symanzik's analysis[4], we express the σ-functionals integrands in (6) as functionals defined on the Feynman-Kac-Wiener space of Random paths by making use of the well known random path representation for the non-relativistic propagator of a particle of mass μ_o in the presence of the external random gaussian potential $\sigma(x)$:[7]

$$(-\Delta+\mu_o^2-2i\sigma)^{-1}(x,y) = \int_0^\infty d\zeta \; G(x,y,\sigma)(\zeta) \tag{7}$$

$$\ell og\mathrm{Det}(-\Delta+\mu_o^2-2i\sigma) = -\int_0^\infty \frac{d\zeta}{\zeta}\int d^Dx \; G(x,x,\sigma)(\zeta) \tag{8}$$

where the non-relativistic propagator is given by

$$G(x,y,\sigma)(\zeta) = \int d\mu[w_{xy}^{(\zeta)}] \; e^{i\int d^Dz \; \sigma(z) j(w_{xy}^{(\zeta)}(z))} \tag{9}$$

with the Feynman-Kac-Wiener path measure

$$d\mu[w_{xy}^{(\zeta)}] = (\prod_{\substack{0<\alpha<\zeta \\ w(0)=x \\ w(0)=y}} dw[\alpha])\exp\{-\frac{1}{2}\int_0^\zeta (\frac{dw}{d\alpha})^2 - \frac{1}{2}\mu_o^2\zeta\} \tag{10}$$

and the (random) world-line currents defined by

$$j(w_{xy}^{(\zeta)})(z) = \int_0^\zeta \delta^D(z-w_{xy}^{(\zeta)}(\alpha))d\alpha \tag{11}$$

So, we obtain the proposed reformulation of $\lambda\phi^4 0(N)$- theory as a theory of random paths $\{w_{xy}^{(\zeta)}(\alpha)\}$ in the presence of a random gaussian potential

$$\begin{aligned} Z[J^a(x)] = \int d\mu[\sigma].\exp\{\frac{N}{2}\int_0^\infty \frac{d\zeta}{\zeta}\int d^Dx \int d\mu[w_{xx}^{(\zeta)}] \\ \exp(i\int d^Dz \; \sigma(z) j(w_{xx}^{(\zeta)})(z)\}\exp\{\frac{1}{2}\int d^Dx \, d^Dy \sum_{a=1}^N J_a(x)[\int_0^\infty d\zeta \int d\mu[w_{xy}^{(\zeta)}] \\ \exp(i\int d^Dz \; \sigma(z) j(w_{xy}^{(\zeta)})(z))]\delta_{ab}.J_b(y)\} \end{aligned} \tag{12}$$

We shall use the random path formulation (12) to analyse the correlation functions of the $\lambda\sigma^4$ theory. As a useful remark, we note by using (12) that the general k-point (bare) correlation function possesses the general structure

$$<\Phi_{i_1}(x_1)\ldots\Phi_{i_k}(x_k)>_\Phi =$$

$$\begin{cases} 0 & \text{if } k=2j+1 \\ \displaystyle\sum_{\substack{(2j+1)!! \\ \ell\text{-pairings}}} <\Phi_{i_1}(x_{\ell_1})\Phi_{i_2}(x_{\ell_2})>_\Phi \ldots <\Phi_{i_{k-1}}(x_{\ell_{2j-1}})\Phi_{i_k}(x_{\ell_{2j}})>_\Phi & \text{if } k=2j \end{cases} \quad (13)$$

where the quantum averages $<\ >_\Phi$ in (13) are defined by the $\lambda\phi^4$ partition functional $Z[0]$ (see eq.(1) with $J^a(x)\equiv 0$).

Because of this result, we have solely to study the properties of the 2-point correlation function

$$<\Phi_{i_1}(x_1)\Phi_{i_2}(x_2)>_\Phi = \delta_{i_1 i_2} < \int_0^\infty d\zeta\, d\mu[w^{(\zeta)}_{x_1 x_2}].\exp\{i\int d^D z\ \sigma(z).j(w^{(\zeta)}_{x_1 x_2})(z)\}$$

$$\exp\{\frac{N}{2}\int_0^\infty \frac{d\zeta}{\zeta}\int d^D x\int d\mu[w^{(\zeta)}_{xx}].\exp\{i\int d^D z\ \sigma(z) j(w^{(\zeta)}_{xx}(z)\}\}>_\sigma \quad (14)$$

Let us evaluate the σ-functional averages $<\ >_\sigma$ in eq.(14) (see eq. (4) and eq. (5)). For this task we expand the "close path term" in powers of N. Explicitly

$$<\Phi_{i_1}(x)\Phi_{i_2}(y)> = \delta_{i_1 i_2}\sum_{k=0}^\infty (\frac{N}{2})^k \{\prod_{\ell=1}^k \int_0^\infty \frac{d\zeta_\ell}{\zeta_\ell}\int d^D x_\ell$$

$$\int d\mu[w^{(\zeta_\ell)}_{x_\ell x_\ell}].\int_0^\infty d\zeta.\int d\mu[w^{(\zeta)}_{xy}].<\exp\{i\sum_{\ell=1}^k\int d^D z_\ell\ \sigma(z_\ell).j(w^{(\zeta_\ell)}_{x_\ell x_\ell})(z_\ell)$$

$$+ i\int d^D z\ \sigma(z) j(w^{(\zeta)}_{xy})\ (z)\}>_\sigma \quad (15)$$

and since the σ-average in eq.(15) is of the gaussian type we can perform it exactly. The result reads:

$$<\Phi_{i_1}(x)\Phi_{i_2}(y)>_\Phi = \delta_{i_1 i_2}\sum_{k=0}^{\infty}\left(\frac{N}{2}\right)^k\left\{\prod_{\ell=1}^{k}\int_0^\infty\frac{d\zeta_\ell}{\zeta_\ell}.\int d^D x_\ell\int d\mu[w^{(\zeta_\ell)}_{x_\ell x_\ell}]\right.$$

$$\int_0^\infty d\zeta.\int d\mu[w^{(\zeta)}_{xy}].\exp\{-\frac{\lambda}{4}[(2.\sum_{\ell\neq\ell'}^{k}\int_0^{\zeta_\ell}d\alpha_\ell.\int_0^{\zeta_{\ell'}}d\alpha_{\ell'}.\ \delta^{(D)}(w^{(\zeta_\ell)}_{x_\ell x_\ell}(\alpha_\ell) -$$

$$w^{(\zeta_{\ell'})}_{x_{\ell'}x_{\ell'}}(\alpha_{\ell'})) + (\sum_{\ell=\ell'}^{k}\int_0^{\zeta_\ell}d\alpha_\ell\int_0^{\zeta_\ell}d\alpha_{\ell'}\ \delta^{(D)}(w^{(\zeta_\ell)}_{x_\ell x_\ell}(\alpha_\ell)-w^{(\zeta_\ell)}_{x_\ell x_\ell}(\alpha_{\ell'})))$$

$$+ (2.\sum_{\ell=1}^{k}\int_0^{\zeta_\ell}d\alpha_\ell.\int_0^{\zeta}d\alpha\ \delta^{(D)}(w^{(\zeta_\ell)}_{x_\ell x_\ell}(\alpha_\ell)-w^{(\zeta)}_{xy}(\alpha))) +$$

$$+ (\int_0^{\zeta}d\alpha\int_0^{\zeta}d\alpha'\ \delta^{(D)}(w^{(\zeta)}_{xy}(\alpha)-w^{(\zeta)}_{xy}(\alpha'))]\} \tag{16}$$

The above expression is the two-point correlation function of the $\lambda\phi^4$-0(N)- theory expressed as a system of interacting random paths with a repulsive self-interaction at these points where they crosses themselves. It is instructive to point out that (16) is related to the continuous version of the Aizenman Z^D-lattice two--point correlation function.

Now we can offer a topological explanation for the theory triviality phenomenon for D>4. At first, we note that the correlation function (17) will differ from the free one, namely

$$<\Phi_{i_1}(x)\ \Phi_{i_2}(y)>_{FREE} = \delta_{i_1 i_2}(\int_0^\infty d\zeta d\mu[w^{(\zeta)}_{xy}]) \tag{17}$$

if the path intersections implyed by the delta functions in (16) are non-empty sets in the R^D space-time. We intend to argument that these intersection sets are empty for space-time with dimensionality greater than four. At first we recall some well-known concepts of topology[8]: the topological Hausdorff dimension of a set A embedded in R^D is d (with d being a real number) if the minimum number of D-dimensional spheres of radius γ needed to cover it, grows like γ^{-d} when $r \to 0$. The rule for (generical) intersections for sets A and B (both are embeddeds in R^D) is given by

$$d(A \cap B) = d(A) + d(B) - D \tag{18}$$

where a negative Hausdorff dimension means no (generical) intersection or equivalently the set $A \cap B$ is empty.

As is well know the Hausdorff dimension of the random paths in (16) is 2.[8,9] A direct application of the rule (18) gives us that the intersection sets in (16) possesses a Hausdorff dimension 4-D. So, for D>4 these sets are empty and leading to the triviality phenomenon (see eq.(17)).

Next, we consider the "Polymer" limit $N \to 0$.[3,5,6] As the number of field components is displayed as a parameter in the model's correlation functions, we can take straightforwardly the above limit. For instance:

$$\langle \Phi_{i_1}(x) \Phi_{i_2}(y) \rangle_{\Phi}^{(N \to 0)} = \int_0^\infty d\zeta \int d\mu [w_{xy}^{(\zeta)}] e^{-\frac{\lambda}{4} \int_0^\zeta d\alpha \int_0^\zeta d\alpha' \, \delta^{(D)}(w_{xy}^{(\zeta)}(\alpha) - w_{xy}^{(\zeta)}(\alpha'))} \tag{19}$$

We note that the (19) is nothing more than the probability of a polymer $\{w_{xy}^{(\zeta)}(\alpha)\}$ of lenght α starting at the point x Hitts the point y with the self-suppressing Edward interaction[5,6].

Finally we make some comments on the analyses of the divergencies in the random path expression (16) for $D \leq 4$. As a first observation we note that all the path integrals involved in (16) can be exactly evaluated by making a power series in λ_0.[5,10] The resulting proper-times ζ integrals will in general be divergents. By using a regularization (such as a cut off for small proper-times) one can show that the divergencies can be absorbed by a renormalization of the bare mass μ_0 and the action path term in (16) (or equivalently, a wave-function and λ_0-coupling renormalization in the field formulation eq.(1)), closer to the renormalization "polymer" procedure exposed in [5].

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