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# ON THE SOLUTIONS OF THE HEAT EQUATION

by

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Abstract

It is shown, for a wide class of operators, that the solution to the corresponding heat equation may be obtained as a series. This is accomplished by the inverse Mellin transform of the Green function of the complex power operator.

Key-words: Heat equation; Mellin transform; Solution.

A renewed interest on the heat equation was observed these last years. Among the several retombées is the regularization of determinants of differential operators that arrives in the Feynman path-integral approach to Quantum Field Theory [1,6]. These determinants are related to the generalized zeta function, on which informations can be obtained in some cases from the solution of the heat equation,

$$\frac{d}{dt} F(x,y,t) = AF(x,y,t) \quad , \tag{1}$$

where t is a (time) parameter and x and y are points of a D-dimensional space-time Euclidean compact manifold without boundary; the operator A acts on the x-variable. For more generality, A may be taken as a pseudo-differential operator [2,3].

Under some conditions the power operator $A^s$ (s complex) may be properly defined. In this case, the Green's function of $A^s$, Z(s,x,y) is related to the solution of eq. (1) by a Mellin transform,

$$Z(s,x,y) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} F(x,y,t) \quad . \tag{2}$$

We show in the following that in situations when it is possible to have a solution of the fundamental equation $A^s Z = \delta(x-y)$, we can always obtain a general solution of eq. (1) by an inverse Mellin transform,

$$F(x,y,t) = \int_{+\infty}^{-\infty} \frac{d\,\mathrm{Im}s}{2i\pi}\, t^{-s}\Gamma(s) Z(s,x,y) \quad , \tag{3}$$

provided $Z(s,x,y)$ can be extended to the complex s-plane. The s-integration in (3) goes parallel to the imaginary axis, and Re(s) must belong to the analyticity domain of $Z(s,x,y)$.

As an illustration of the method, let us take A to be the Laplacian operator, $A = -\partial^2$ in D dimensions. The Green function of $A^k$, for real integer positive k, $Z_k(x,y)$, is well known [4]. Starting from this result we perform the extension from integers k to complex s-values, obtaining a meromorphic function,

$$Z(s,x,y) = (-1)^s \frac{e^{i\pi\frac{D}{2}}\Gamma(\frac{D}{2}-s)(P+io)^{-\frac{D}{2}+s}}{4^s\Gamma(s)\pi^{\frac{D}{2}}} \quad , \; Re(s) < \frac{D}{2} \quad , \qquad (4)$$

where P is the quadratic form $-\sum_{i=1}^{D}(x_i - y_i)^2 \equiv -(x-y)^2$.
The singularities of the factor $\Gamma(s)Z(s,x,y)$ in the integrand of eq. (3) are poles located at the points $s = j+\frac{D}{2}$, $j=0,1,2,\ldots$ (coming from $\Gamma(\frac{D}{2}-s)$) and $s = -j$ (coming from $(P+io)^{-\frac{D}{2}+s}$). The integration contour $C_0$ in fig. 1) may be displaced to the right, picking up successively the contributions from the poles at $s = j+\frac{D}{2}$, from $j=0$ up to $j=\infty$. The procedure gives the result,

$$F(x,y,t) = (-1)^D (4\pi t)^{-\frac{D}{2}} \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{P+io}{4t}\right)^j = (-1)^D (4\pi t)^{-\frac{D}{2}} \exp\left[-(x-y)^2/4t\right] \qquad (5)$$

which is the well known solution to the classical heat equation.

The method may be extended formally to the case of a general pseudo-differential operator of order $\underline{m}$, defined on the D-dimensional compact manifold. When $Re(s) < -\frac{D}{m}$, $A^s$ has a continuous Kernel $K(s,x,y)$ [5] which is related to the Green function $Z(s,x,y)$ throught $K(s,x,y) = Z(-s,x,y)$. Then eq. (3) is rewritten as,

$$F(x,y,t) = \int_{+\infty}^{-\infty} \frac{d\,\mathrm{Im}\,s}{2i\pi} t^{-s}\Gamma(s)K(-s,x,y) \quad , \tag{6}$$

In the approximation used in [5] to construct the power operator $A^s$, the diagonal elements of the Kernel, $K(s,x,x)$ extend to meromorphic functions of s, whose poles are located at $s = \frac{j-D}{2}$, $j = 0,1,2,...$, with known residues. The elements $K(s,x,y)$ $x \neq y$, extend to entire functions.

To proceed as in the proceeding case, we note that there are double poles in the integrand of eq. (3) for real non-positive integer values of s, $s = -\ell$, $\ell = 0,1,2,...$. The other poles are simple ones (fig. 2). The integration contour $C_0$ (fig. 2) may by displaced to the left, picking up successively the simple poles as before. When the displaced integration contour meets a double pole (the first one at $s=0$), a slight different procedure is needed: writing $\Gamma(s)K(-s) \underset{s\sim-\ell}{\approx} \frac{\phi(s)}{(s+\ell)^2}$, we perform an integration by parts, obtaining an integral of the type,

$$\int_C \frac{d\,\mathrm{Im}\,s}{2i\pi} t^{-s} \frac{-(\ell n t)\phi(s) + \frac{d\phi}{ds}}{s+\ell}$$

But from [5] the residues of $K(-s)$ at $s=-\ell$ vanish, which suppress the term in $\ell n t$ in the integral above. Thus the diagonal elements of the solution of the general heat equation are always expressible as the following series:

$$F(t,x,x) = -\sum_{\ell=0}^{\infty} t^{-\ell} \left.\frac{d\phi}{ds}\right|_{s=-\ell} - \sum_{j=0}^{\infty} t^{-\frac{D-j}{m}} \Gamma\left(\frac{D-j}{m}\right) R_j \tag{7}$$

$$\frac{D-j}{m} \neq 0,-1,-2,...$$

where $R_j$ is the residue of $K(-s,x,x)$ at $s = \frac{D-j}{\underline{m}}$ .

Some comments are in order. The solution given in eq. (7) if we take $A = -\partial^2$ does not coincide with the known solution for the Laplacian which we have reproduced above (the case $F(t,x,x)$ corresponds to take $y = 0$ in eq. (5). This is due to the fact that the analytic structure of the Kernel as given in [5] is rather different from that of the exact Kernel obtained from the exact Green function of our first example. Actually this is not surprising since in [5] an approximation is made to the power operator $A^s$. Even so, this approximated Kernel reproduces *exactly* the axial anomaly term for $QCD_4$[6,7]. Perhaps this curious result is linked to the fact that the radiactive corrections do not contribute to the axial anomaly.

The method described here may be used for precise calculations in the case of order $\underline{m}$ differential operators of the general form

$$\sum_{\alpha_1+\ldots+\alpha_n \leq \underline{m}} A_{\alpha_1\ldots\alpha_n}(x) \frac{\partial^{\alpha_1+\ldots+\partial_n}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}} .$$

These will be the subject of a forthcoming paper.

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## Figure Captions

Fig. 1 - Poles of $\Gamma(s)K(-s)$ for the case where the A is Laplacian. $K(s)$ is the exact Kernel of $A^s$.

Fig. 2 - Poles of $\Gamma(s)K(-s)$ for a general pseudo differential operator A; $K(s)$ is the approximate Kernel of $A^s$ as given in ref. [5].

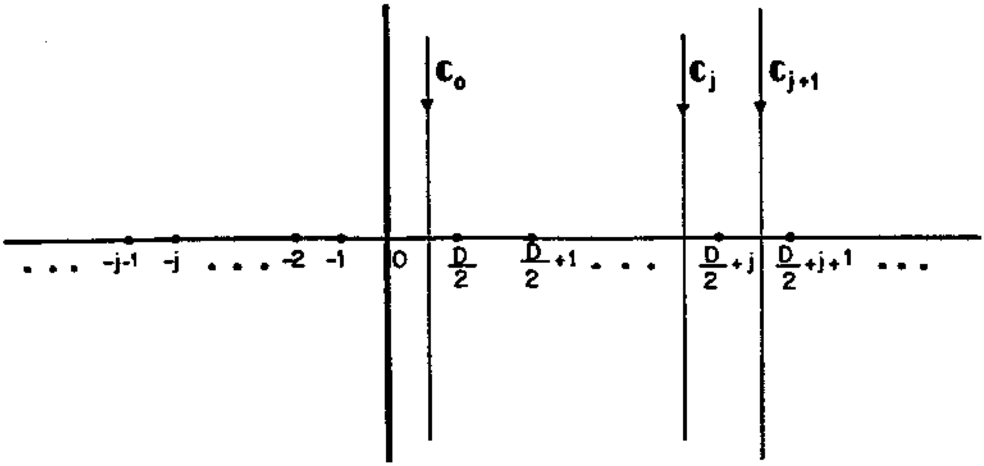
$\mathbb{C}_0$
$\mathbb{C}_j$
$\mathbb{C}_{j+1}$
... -j-1 -j ... -2 -1
0
$\frac{D}{2}$
$\frac{D}{2}+1$ ...
$\frac{D}{2}+j$
$\frac{D}{2}+j+1$ ...

FIG.1

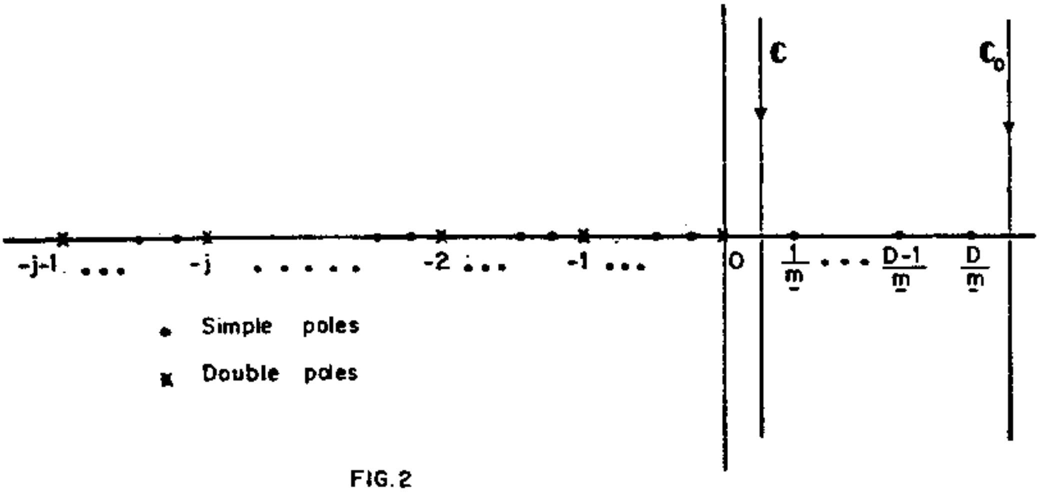
$\mathbb{C}$
$\mathbb{C}_0$
-j-1 ... -j ..... -2 ... -1 ...
0
$\frac{1}{m}$ ... $\frac{D-1}{m}$ $\frac{D}{m}$
• Simple poles
x Double poles

FIG.2

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