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DOUBLE GROUPS OF POINT GROUPS

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ABSTRACT

The representations of point groups appropriate to spin-$\frac{1}{2}$ particles are usually determined by considerations of a group of double the order of the point group under consideration, known as the double group. An alternative definition for double groups is given which makes use of the theory of central extensions and presentations of the point groups.

Key-words: Double finite groups.

It is sometimes said that the j half-integer representations of SU(2) are "double valued" representations of SO(3), allowed because of the nature of the measurement process. We prefer to say that the j half-integral representations are faithful representations of SU(2). Moreover, j half-integer defines the isomorphism $SU(2)/Z_2 \simeq SO(3)$, where Z_2 is generated by the element of SU(2) which corresponds to the rotation $R(2\pi, \bar{n})$ and therefore it is the group of the center of SU(2). Also, since every finite subgroup G* of SU(2) must obviously contain Z_2 it must also satisfy $G^*/Z_2 \simeq G$ $(G < SO(3))$.

Knowing G, the group associated with the geometry of a system, the determination of G* is performed by the solution of the problem of a central extension of the group Z_2 by G. This problem has not a unique solution and in order to solve it, Altmann[1] has developed a method based on the structure of the poles of the group G which allows to determine the factor system of the central extension and consequently G*.

Here we show that for point groups it is possible to obtain a solution of G* without defining the factor system of the central extension and that this solution is the so called double group of the group G.

A group G of order $|G|$ is finitely presented if it has a presentation $< x \mid R >$, that is, one in which the group can be specified by a finite set of generators $x = x_1, x_2, \ldots, x_r$ $(r<|G|)$ and a finite set of relations between them, $R_\ell(x_1,x_2,\ldots,x_r) = E$, where E is the unit element of G.

The defining presentations of the groups C_n, D_n, A_n and S_n are[2]

$$C_n = \langle x \parallel x^n = E \rangle$$

$$D_n = \langle x_1, x_2 \parallel x_1^2 = x_2^2 = (x_1 x_2)^n = E \rangle$$

$$A_n = \langle x_1, x_i \parallel x_1^3 = x_i^2 = (x_{i-1} x_i)^3 = (x_j x_k)^2 = E \rangle ,$$

where $1 < i \leq n-2$, $|j-k| > 1$;

$$S_n = \langle x_i \parallel x_i^2 = (x_i x_{i+1})^3 = (x_i x_j)^2 = E \rangle ,$$

where $1 \leq i < n$, $|i-j| > 1$.

Leaving in mind the isomorphisms $T - A_4$, $O - S_4$ and $I - A_5$, we have all the proper point groups given by the preceding presentations.

Now we show that if G is a subgroup of SO(3) presented by

$$G = \langle x_i \parallel R_\ell = E \rangle \quad , \tag{1}$$

then the group presented by

$$G^* = \langle y_i \parallel R_\ell = z \ , \ z^2 = 1 \rangle \quad , \tag{2}$$

where $y_i = (x_i, 1)$, $z = (1,-1)$ and $1 = (1,1)$, is the solution of the central extension of Z_2 by G called double group of G. First we must show that given the group H defined by

$$H = \langle y_i \parallel R_\ell = z_\ell \ , \ z_\ell \in Z_2 \rangle \quad , \tag{3}$$

where for at least one R_ℓ, z_ℓ takes the value z, is such that $H/Z_2 - G$,which is a necessary condition for H to be the double group of G.

Let $h \in H$ be given by

$$h = y_1^{k_1} y_2^{k_2} \cdots y_1^{\ell_1} y_2^{\ell_2} \cdots ,$$

where $(k_i, \ell_i, \ldots) < |x_i|$. We can define a mapping φ such that

$$\varphi : h \to g \quad \text{and} \quad \varphi : hz \to g \quad ,$$

where $$g = x_1^{k_1} x_2^{k_2} \ldots x_1^{\ell_1} x_2^{\ell_2} \ldots$$

is an element of G. Then, $\varphi : H = G$ and kernel $\varphi = Z_2 = 1, z$. Moreover, since the relations between the generators x_i of G and those between the generators y_i of H differ only in an element of the group Z_2 of the center of H, we have that, for $g_i g_j = g_k$ in G , $h_i h_j Z_2 = h_k Z_2$ in H/Z_2 and therefore $H/Z_2 - G$.

Second, assume that we have H given by (3). If we want it to be isomorphic to the double group G* of G we must impose that those relations $R_\ell(x_i) = E$ which define the involutions in G, take the form $R_\ell(y_i) = z$ in H. Since our presentations define at least one element of each conjugacy class of the elements of order 2 in G, the condition forces H to contain only one involution, the element z. Moreover, if the order n of an element $g \in G$ is even, then the element given by $g^{n/2}$ is an involution of G and therefore

$$(g^{n/2}, 1)^2 = (g^{n/2}, -1)^2 = z \quad .$$

This implies that

$$(g,1)^n = (g,-1)^n = z \quad .$$

So, the condition also doubles in H the order of the corresponding element of even order in G .

Third, it is clear that the relations $R_\ell(y_i) = z_\ell$ for the elements of odd order, does not define different central extensions $G^*/Z_2 - G$, but merely equivalent ones since every presentation arising from different values of z_ℓ, can take the form $R_\ell = z \ \forall \ \ell$, through the equivalence relation $y_i \to y_i z_i$ for convenient $z_i \in Z_2$. This argument completes the proof.

Finally, we should like to mention that equation (2) also gives the double groups of improper groups.

The improper groups containing the inversion as a symmetry element are all direct products with C_i. Given $G = G_1 \times C_i$, where $G_1 < SO(3)$, it has been shown[3] that the factor system of the central extension

$$(G_1 \times C_i)^*/Z_2 - G_1 \times C_i$$

is such that

$$f(i,g) = f(g,i) = f(i,i) = 1 \ , \ g \in G_1 \ .$$

This property directly leads to

$$(G_1 \times C_i)^* - G_1^* \times C_i^* \ ,$$

where $C_i^* - C_i$ and G_1^* is given by (2).

The groups $C_{n,h} - C_n \times C_i$, $D_{2n+1,d} - D_{2n+1} \times C_i$, $D_{2n,h} - D_{2n} \times C_i$, $T_h - T \times C_i$, $O_h - O \times C_i$ and $I_h - I \times C_i$ are included in this case.

When G is an improper group which does not contain the inversion explicitly, it can be written as a coset expansion[4] $H + igH$, where H is a subgroup of SO(3) and $g \notin H$ is a proper rotation of even order. Moreover, in this case there always exists a group G', also a subgroup of

SO(3), isomorphic to G, such that $G' = H + gH$. Now, when dealing with half-integral eigenfunctions of the angular momentum, the inversion is represented by a unit matrix and therefore, G and G' will have the same double valued representations. Then we can conclude that G* is isomorphic to G'* and G'* is also given by (2). Thus we cover all the remaining point groups $C_{n,v} - D_n$, $D_{2n,d} - D_{4n}$, $D_{2n+1,h} - D_{4n+2}$ and $T_d - O$.

As a final remark let us stress that the convention usually adopted to label the classes of double groups in character tables, is misleading as it suggests an incorrect interpretation of the order of the elements whenever this order is even.

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