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LAGRANGIAN PROCEDURES FOR HIGHER ORDER FIELD EQUATIONS

by

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ABSTRACT

We present in a pedagogical way a Lagrangian procedure for the treatment of higher order field equations. We build the energy-momentum tensor and the conserved density current. In particular we discuss the case in which the derivatives appear only in the invariant D'Alembertian operator. We discuss some examples. We quantize the fields and construct the corresponding Hamiltonian which is shown not to be positive definite. We give the rules for the causal propagators.

Key-words: Higher order equations; Lagrangian procedures; Field theory.

1 INTRODUCTION

Ever since the advent of differential equations for the description of physical systems, the consideration of higher order field equations has always been present in the mind of physicists. It is almost impossible to mention all references to earlier works on the subject, but we would like to point out that a Lagrangian treatment was already present in Courant-Hilbert's book [1], and the classical retarded Green function was constructed in reference [2].

Nowadays the subject is adquiring increasing importance due to the consideration of gravity theories with Lagrangian containing terms quadratic in the curvature tensor [3]. Furthermore, supersymmetry in higher dimensions leads to higher order equations, so that dimensionality of space-time could be related to the order of the field equations [4] [5]. Also in this context, it might turn out to be impossible to get complete conformal invariance with fields obbeying only second order wave equations [6].

It seems then convenient to attack the problem with canonical methods, trying to understand and overcome, if possible, the difficulties one encounters [7].

It is for these reasons that we here develope a general formalism for higher order equations starting from a Lagrangian and building up from it, the canonical tensors. In this respect we don't pretend to be fully original, but we rather try to systematize in an easy way, the procedures that can be followed for the development of the theory.

For an alternative, more mathematical, viewpoint see reference [8].

We should also mention that it is possible to follow Schwinger's Action integral methods [9] the canonical tensors coinciding, up to divergences, with those obtained here. The equations of motion are, of course, the same.

Lastly we would like to point out the general appearance of negative energy states, which should be related to an indefinite metric in the "Hilbert space" of states, for whose treatment several references can be given (see for example [10]).

It should be pointed out also that the whole atittude and philosophy regarding the usual S-matrix problem should be changed in theories of higher derivatives. This is one of the main problems which has to be clarified in the near future if one intends to go ahead with higher order equations. We intend to treat this problem in a forthcomming paper.

2 CANONICAL TENSORS

We start with a Lagrangian function of a scalar field ϕ and of its first m derivatives, in an n-dimensional space-time.

$$\mathcal{L} = \mathcal{L}(\phi, \partial\phi, \ldots, \partial^m\phi) \tag{2.1}$$

The principle of least action A, allows us to write

$$A = \int d^n x \, \mathcal{L} \, , \tag{2.2}$$

$$\delta A = \int d^n x \delta \mathcal{L} = 0 \tag{2.3}$$

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi} \delta \partial_\alpha \phi + \ldots \frac{\partial \mathcal{L}}{\partial \partial_{\alpha_1} \ldots \partial_{\alpha_m} \phi} \delta (\partial_{\alpha_1} \ldots \partial_{\alpha_m} \phi) \tag{2.4}$$

After integrating by parts the variation of the derivatives of the field ϕ, (2.3) and (2.4) lead to the following Euler equation:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi} + \partial_{\alpha_1} \partial_{\alpha_2} \frac{\partial \mathcal{L}}{\partial \partial_{\alpha_1} \partial_{\alpha_2} \phi} - \ldots \pm \partial_{\alpha_1} \ldots \partial_{\alpha_m} \frac{\partial \mathcal{L}}{\partial \partial_{\alpha_1} \ldots \partial_{\alpha_n} \phi} = 0 \tag{2.5}$$

Similarly, we can deduce the generalized Nöther theorem.

$$\begin{aligned} \delta \mathcal{L} = \partial_\alpha \Bigg\{ & \left(\frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi} - \partial_\beta \frac{\partial \mathcal{L}}{\partial \partial_\alpha \partial_\beta \phi} + \ldots \right) \delta \phi + \\ & \left(\frac{\partial \mathcal{L}}{\partial \partial_\alpha \partial_\beta \phi} - \partial_\gamma \frac{\partial \mathcal{L}}{\partial \partial_\alpha \partial_\beta \partial_\gamma \phi} + \ldots \right) \delta \partial_\beta \phi + \\ & \left(\frac{\partial \mathcal{L}}{\partial \partial_\alpha \partial_\beta \partial_\gamma \phi} - \ldots \right) \delta \partial_\beta \partial_\gamma \phi + \ldots \Bigg\} \end{aligned} \tag{2.6}$$

In particular we construct the energy-momentum tensor $T^{\mu\nu}$ by considering inifinitesimal translation:

$$\delta x^\mu = \varepsilon^\mu \quad ; \quad \delta \mathcal{L} = \partial_\mu \mathcal{L} \varepsilon^\mu \quad ; \quad \delta \phi = \partial_\mu \phi \varepsilon^\mu$$

$$\delta \partial_\alpha \phi = \partial_\mu \partial_\alpha \phi \varepsilon^\mu \ , \ \ldots \text{etc.} \tag{2.7}$$

Replacing (2.7) in (2.6) we get:

$$\partial_\alpha T^{\alpha\mu} = 0 \tag{2.8}$$

where:

$$T^{\alpha\mu} = \left(\frac{\partial\mathcal{L}}{\partial\partial_\alpha\phi} - \partial_\beta \frac{\partial\mathcal{L}}{\partial\partial_\alpha\partial_\beta\phi} + \partial_\beta\partial_\gamma \frac{\partial\mathcal{L}}{\partial\partial_\alpha\partial_\beta\partial_\gamma\phi} - \ldots\right) \partial^\mu\phi +$$
$$\left(\frac{\partial\mathcal{L}}{\partial\partial_\alpha\partial_\beta\phi} - \partial_\gamma \frac{\partial\mathcal{L}}{\partial\partial_\alpha\partial_\beta\partial_\gamma\phi} + \ldots\right) \partial_\beta\partial^\mu\phi +$$
$$\left(\frac{\partial\mathcal{L}}{\partial\partial_\alpha\partial_\beta\partial_\gamma\phi} - \ldots\right) \partial_\beta\partial_\gamma\partial^\mu\phi + \ldots - \eta^{\alpha\mu}\mathcal{L} \tag{2.9}$$

This tensor is not necessarily symmetric but it can be symmetrized following Belinfante's procedure[11]. Anyway, as the symmetry is broken by divergence terms, the total energy-momentum vector is well defined by (2.9).

$$P^\mu = \int d^n x T^{0\mu} \tag{2.10}$$

When ϕ is complex and the Lagrangian is phase-invariant, one is led from (2.6) to the conserved current: ($\delta\phi = i\varepsilon\phi$; $\delta\phi^* = -i\varepsilon\phi^*$)

$$j^\alpha = i\varepsilon \left\{ \left(\frac{\partial\mathcal{L}}{\partial\partial_\alpha\phi} - \partial_\beta \frac{\partial\mathcal{L}}{\partial\partial_\alpha\partial_\beta\phi} + \ldots\right) \phi + \right.$$
$$\left. \left(\frac{\partial\mathcal{L}}{\partial\partial_\alpha\partial_\beta\phi} - \partial_\gamma \frac{\partial\mathcal{L}}{\partial\partial_\alpha\partial_\beta\partial_\gamma\phi} + \ldots\right) \partial_\beta\phi + \ldots - \text{c.c.} \right\} \tag{2.11}$$

3 FUNCTIONS OF ITERATED D'ALEMBERTIAN

We shall discuss special case in which the Lagrangian is a function of the derivatives of ϕ, only through the invariant D'Alembertian operator:

$$\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta \tag{3.1}$$

We define

$$\phi^{(s)} = \Box^s \phi \qquad (\phi^{(o)} \equiv \phi) \tag{3.2}$$

$$\mathcal{L} = \mathcal{L}(\phi^{(s)}) \qquad (s = 0,1,\ldots M)$$

The principle of least action now leads (cf. (2.4), (2.5)), to the Euler equation:

$$\frac{\partial\mathcal{L}}{\partial\phi} + \Box\,\frac{\partial\mathcal{L}}{\partial\phi^{(1)}} + \Box^2\,\frac{\partial\mathcal{L}}{\partial\phi^{(2)}} + \ldots = 0$$

Or:

$$\sum_{s=0}^{M} \Box^s\,\frac{\partial\mathcal{L}}{\partial\phi^{(s)}} = 0 \tag{3.3}$$

Similarly, we have the Nöther theorem:

$$\delta\mathcal{L} = \partial_\alpha \sum_{s,t=0}^{M}\left\{\Box^s\,\frac{\partial\mathcal{L}}{\partial\phi^{(s+t+1)}}\,\partial^\alpha\delta\phi^{(\varepsilon)} - \partial^\alpha\,\Box^s\,\frac{\partial\mathcal{L}}{\partial\phi^{(s+t+1)}}\,\delta\phi^{(t)}\right\} \tag{3.4}$$

(Compare with (2.6)).

From (3.4) we deduce, using (2.7), the energy-momentum tensor

$$T^{\alpha\mu} = \sum_{s,t=0}^{M} \left\{ \Box^s \frac{\partial \mathcal{L}}{\partial \phi^{(s+t+1)}} \partial^\alpha \partial^\mu \phi^{(t)} - \partial^\alpha \Box^s \frac{\partial \mathcal{L}}{\partial \phi^{(s+t+1)}} \partial^\mu \phi^{(t)} \right\} - \eta^{\alpha\mu} \mathcal{L} \tag{3.5}$$

(Compare with (2.9)).

This tensor can again be symmetrized following Ref. [11]. It is easily seen that

$$\partial_\mu T^{\mu\nu} = 0 \tag{3.6}$$

The conserved current takes now the form

$$j^\mu = i\varepsilon \left\{ \sum_{s,t=0}^{M} \left(\Box^s \frac{\partial \mathcal{L}}{\partial \phi^{(s+t+1)}} \partial^\mu \phi^{(t)} - \partial^\mu \Box^s \frac{\partial \mathcal{L}}{\partial \phi^{(s+t+1)}} \phi^{(t)} - ch \right\} \tag{3.7}$$

where ch means hermitian conjugate

4 EXAMPLES

a) Let us begin with the simple example of the usual Klein-Gordon equation, treated from the point of view of a higher order Lagrangian.

$$\mathcal{L} = -\frac{1}{2} \phi^* \Box \phi - \mu^2 \phi^* \phi - \frac{1}{2} \phi \Box \phi^* \tag{4.1}$$

from which

$$\frac{\partial\mathcal{L}}{\partial\phi} = -\mu^2\phi^* - \frac{1}{2}\Box\phi^* ; \frac{\partial\mathcal{L}}{\partial\phi^{(1)}} = -\frac{1}{2}\phi^* \tag{4.2}$$

giving the equations of motion

$$-\mu^2\phi^* - \frac{1}{2}\Box\phi^* + \Box\left(-\frac{1}{2}\phi^*\right) = 0 \tag{4.3}$$

For the energy-momentum tensor:

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\phi^{(1)}}\partial^\mu\partial^\nu\phi + \frac{\partial\mathcal{L}}{\partial\phi^{(1)*}}\partial^\mu\partial^\nu\phi^* - \frac{\partial^\mu\partial\mathcal{L}}{\partial\phi^{(1)}}\partial^\nu\phi - \frac{\partial^\mu\partial\mathcal{L}}{\partial\phi^{(1)*}}\partial^\nu\phi^* - \eta^{\mu\nu}\mathcal{L}$$

$$= -\frac{1}{2}\phi^*\partial^\mu\partial^\nu\phi + \frac{1}{2}\partial^\mu\phi^*\partial^\nu\phi \tag{4.4}$$

$$-\frac{1}{2}\phi\partial^\mu\partial^\nu\phi^* + \frac{1}{2}\partial^\mu\phi\partial^\nu\phi^* - \eta^{\mu\nu}\mathcal{L}$$

and for the Hamiltonian

$$\mathcal{H} = T^{00} = -\frac{1}{2}\phi^*\ddot{\phi} + \frac{1}{2}\dot{\phi}^*\dot{\phi} - \frac{1}{2}\phi\ddot{\phi}^* + \frac{1}{2}\dot{\phi}\dot{\phi}^*$$

$$-\left(-\frac{1}{2}\phi^*\Box\phi - \frac{1}{2}\phi\Box\phi^* - \mu^2\phi^*\phi\right) =$$

$$= -\frac{1}{2}\phi^*\nabla^2\phi - \frac{1}{2}\phi\nabla^2\phi^* + \dot{\phi}^*\dot{\phi} + \mu^2\phi^*\phi$$

which leads, up to a divergence, to

$$H = |\nabla\phi|^2 + |\dot{\phi}|^2 + \mu^2|\phi|^2 \tag{4.5}$$

The current (3.7) is here the usual one: $j^{\mu} = -i\varepsilon(\phi^*\partial^{\mu}\phi - \partial^{\mu}\phi^*\phi)$.

b) Another example.

$$\mathcal{L} = \frac{1}{2}\Box\phi\Box\phi - \frac{1}{2}\mu^4\phi^2 \tag{4.6}$$

leading to the eq. of motion

$$-\mu^4\phi + \phi^{(2)} = 0 \quad \text{ie:}\ \Box\Box\phi - \mu^4\phi = 0 \tag{4.7}$$

and for the energy-momentum tensor

$$T^{\mu\nu} = \phi^{(1)}\partial^{\mu}\partial^{\nu}\phi - \partial^{\mu}\phi^{(1)}\partial^{\nu}\phi - \eta^{\mu\nu}\mathcal{L} \tag{4.8}$$

The Hamiltonian is given by:

$$\mathcal{H} \equiv T^{00} = \phi^{(1)}\ddot{\phi} - \dot{\phi}^{(1)}\dot{\phi} + \frac{1}{2}\phi^{(1)}\phi^{(1)} + \frac{1}{2}\mu^2\phi^2 \tag{4.9}$$

Using Fourier development for ϕ,

$$\phi(x) = \int dk\, e^{-ikx}\tilde{\phi}(k) \tag{4.10}$$

The equation of motion (4.7) implies:

$$(k^4 - \mu^4)\tilde{\phi} = 0 \tag{4.11}$$

I.e:

$$\tilde{\phi}_{(k)} = \phi_1(k)\delta(k^2-\mu^2) + \phi_2(k)\delta(k^2+\mu^2) \qquad (4.12)$$

And we can see the appearance of two kind of particles. One "normal" with $k^2 = \mu^2$, and another "abnormal", with negative mass square: $k^2 = -\mu^2$. This second particle is a "tachyon"[12].

c) Let us consider a slightly more general example[13]

$$\mathcal{L} = \frac{1}{2}\left(\Box + m_1^2\right)\phi\left(\Box+m_2^2\right)\phi \qquad (4.13)$$

With Euler equations:

$$\frac{\partial\mathcal{L}}{\partial\phi} = \alpha\phi^{(1)} + \beta\phi \quad , \quad \frac{\partial\mathcal{L}}{\partial\phi^{(1)}} = \phi^{(1)} + \alpha\phi$$

$$\alpha = \frac{1}{2}\left(m_1^2+m_2^2\right) \quad , \quad \beta = m_1^2m_2^2 \qquad (4.14)$$

$$\left(\Box+m_1^2\right)\left(\Box+m_2^2\right)\phi = 0 \qquad (4.15)$$

From (3.5):

$$T^{\mu\nu} = (\phi^{(1)}+\alpha\phi)\partial^\mu\partial^\nu\phi - (\partial^\mu\phi^{(1)}+\alpha\partial^\mu\phi)\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$$

$$\mathcal{H} = (\Box\phi+\alpha\phi)\ddot{\phi} - (\Box\dot{\phi}+\alpha\dot{\phi})\dot{\phi} - \mathcal{L} \qquad (4.16)$$

Taking the Fourier transform of ϕ:

$$\phi(x) = \int dk\, e^{-ikx}\tilde{\phi}(k) \quad ; \quad \tilde{\phi}^*(k) = \tilde{\phi}(-k) \qquad (4.17)$$

and using equation (4.15):

$$\left(-k^2+m_1^2\right)\left(-k^2+m_2^2\right)\tilde{\phi}(k) = 0$$

Which means that:

$$\tilde{\phi}(k) = \phi_1(k)\,\delta\left(k^2-m_1^2\right) + \phi_2(k)\,\delta\left(k^2-m_2^2\right)$$

$$= \phi_1(k)\,\delta\left(k_o^2-\omega_1^2\right) + \phi_2(k)\,\delta\left(k_o^2-\omega_2^2\right)$$

$$= \phi_1(k)\,\frac{\delta(k_o+\omega_1)+\delta(k_o-\omega_1)}{2\omega_1} + \phi_2(k)\,\frac{\delta(k_o+\omega_2)+\delta(k_o-\omega_2)}{2\omega_2} \tag{4.18}$$

$$\omega_1 = \sqrt{\vec{k}^2+m_1^2}\ , \quad \omega_2 = \sqrt{\vec{k}^2+m_2^2}$$

We have then:

$$\phi(x) = \int dk\, e^{-i\vec{k}.\vec{r}}\left(\phi_1(\vec{r})\frac{e^{-i\omega_1 t}}{2\omega_1} + \phi_1^*(-\vec{k})\frac{e^{i\omega_1 t}}{2\omega_1} + \phi_2(\vec{k})\frac{e^{-i\omega_2 t}}{2\omega_2} + \phi_2^*(-\vec{k})\frac{e^{*i\omega_2 t}}{2\omega_2}\right) \tag{4.19}$$

The total Hamiltonian is obtained now, replacing (4.19) in (4.16) and integrating over the space variables. The result is:

$$H = \frac{m_1^2-m_2^2}{2}\int dk\,(\phi_1^*\phi_1 - \phi_2^*\phi_2) \tag{4.20}$$

Note the difference in sign between the contributions of the partial fields ϕ_1 and ϕ_2 to the total energy. This is a general feature of higher order equations (see for example Ref. [7]). The quantization can be carried through from (4.20) by imposing Heinsenberg equations of motion:

$$\left[H\ ,\ \phi(x)\right] = -i\dot{\phi}(x) \tag{4.21}$$

Implying for (4.19)

$$\left[H\ ,\phi_1(\vec{k})\right] = -\phi_1\omega_1(\vec{k})\ ;\ \left[H\ ,\phi_2(\vec{k})\right] = -\ \omega_2\phi_2(\vec{k}) \tag{4.22}$$

These commutation relations together with (4.20) give:

$$\left.\begin{aligned} \left[\phi_1(\vec{k}),\phi_1^*(\vec{k}')\right] &= \frac{2\omega_1}{m_1^2-m_2^2}\ \delta(\vec{k}-\vec{k}') \\ \left[\phi_2(\vec{k}),\phi_2^*(\vec{k}')\right] &= -\ \frac{2\omega_2}{m_1^2-m_2^2}\ \delta(\vec{k}-\vec{k}') \end{aligned}\right\} \tag{4.23}$$

With a significative difference in sign. One of the partial fields gives an indefinite metric to the Hilbert space of particle states. (See for example Ref. [10]).

In view of the relations (4.23), we can redefine the fields ϕ_1 and ϕ_2 in such a way that (we take $m_1^2 > m_2^2$):

$$\sqrt{\frac{m_1^2-m_2^2}{2\omega_i}}\ \phi_i \to \phi_i \tag{4.24}$$

Thus we get:

$$H = \int dk\left(\omega_1\phi_1^*\phi_1 - \omega_2\phi_2^*\phi_2\right) \tag{4.25}$$

and

$$\left[\phi_1(\vec{k}),\phi_1^*(\vec{k})\right]=\delta(\vec{k}-\vec{k}') \quad ; \quad \left[\phi_2(\vec{k}),\phi_2^*(\vec{k}')\right]=-\delta(\vec{k}-\vec{k}') \qquad (4.26)$$

5 MANY MASSIVE STATES

Just to establish the pattern we will first consider the following equation of motion:

$$\left(\Box+m_1^2\right)\left(\Box+m_2^2\right)\left(\Box+m_3^2\right)\left(\Box+m_4^2\right)\phi=0 \qquad (5.1)$$

which can be written as:

$$\left(\Box^4+a_3\Box^3+a_2\Box^2+a_1\Box+a_o\right)\phi=0 \qquad (5.2)$$

where

$$a_3=\sum_{i=1}^{4}m_i^2 \;;\; a_2=\sum_{i\neq j}m_i^2m_j^2 \;;\; a_1=\sum_{\substack{i\neq j\neq k\\ i\neq k}}m_i^2m_j^2m_k^2$$

$$a_o=m_1^2m_2^2m_3^2m_4^2 \qquad (5.3)$$

The Lagrangian is:

$$\mathcal{L}=\frac{1}{2}\Box^2\phi\Box^2\phi+\frac{1}{2}a_3\Box\phi\Box^2\phi+\frac{1}{2}a_2\Box\phi\Box\phi+\frac{1}{2}a_1\phi\Box\phi+\frac{1}{2}a_o\phi\phi \quad (5.4)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = a_o \phi + \frac{1}{2} a_1 \phi^{(1)} \quad ; \quad \frac{\partial \mathcal{L}}{\partial \phi^{(1)}} = \frac{1}{2} a_1 \phi + a_2 \phi^{(1)} + \frac{1}{2} a_3 \phi^{(2)}$$

$$\frac{\partial \mathcal{L}}{\partial \phi^{(2)}} = \phi^{(2)} + \frac{1}{2} a_3 \phi^{(1)} \tag{5.5}$$

From (3.5), the Hamiltonian density is seen to be:

$$\mathcal{H} = \left(\frac{\partial \mathcal{L}}{\partial \phi^{(1)}} + \Box \frac{\partial \mathcal{L}}{\partial \phi^{(2)}} \right) \ddot{\phi} - \partial^0 \left(\frac{\partial \mathcal{L}}{\partial \phi^{(1)}} + \Box \frac{\partial \mathcal{L}}{\partial \phi^{(1)}} \right) \dot{\phi} +$$

$$+ \frac{\partial \mathcal{L}}{\partial \phi^{(2)}} \ddot{\phi}^{(1)} - \partial^0 \frac{\partial \mathcal{L}}{\partial \phi^{(2)}} \dot{\phi}^{(1)} - \mathcal{L} \tag{5.6}$$

Using (5.5) and the equation of motion (5.2), we get:

$$\mathcal{H} = -\left(\frac{1}{\Box} a_o + \frac{1}{2} a_1 \right) \phi \ddot{\phi} + \left(\frac{1}{\Box} a_o + \frac{1}{2} a_1 \right) \dot{\phi} \dot{\phi}$$

$$+ \left(\phi^{(2)} + \frac{1}{2} a_3 \phi^{(1)} \right) \ddot{\phi}^{(1)} - \left(\dot{\phi}^{(2)} + \frac{1}{2} a_3 \dot{\phi}^{(1)} \right) \dot{\phi}^{(1)} - \mathcal{L} \tag{5.7}$$

For the Fourier transform of the field, we write:

$$\phi(x) = \int dk\, e^{-ikx} \tilde{\phi}(k) \quad ; \quad \tilde{\phi}^*(k) = \tilde{\phi}(-k) \tag{5.8}$$

$$\tilde{\phi}(k) = \phi_1(k) \delta\left(k^2 - m_1^2\right) + \phi_2(k) \delta\left(k^2 - m_2^2\right) + \phi_3(k) \delta\left(k^2 - m_3^2\right) +$$

$$+ \phi_4(k) \delta\left(k^2 - m_4^2\right) \tag{5.9}$$

Then, as in (4.18), (4.19) we get:

$$\phi(x) = \int dk\, e^{i\vec{k}\vec{r}} \sum_{j=k}^{4} \frac{1}{2\omega_j} \left(\phi_j(\vec{k}) e^{-i\omega_j t} + \phi_j^*(-\vec{k}) e^{i\omega_j t} \right) \tag{5.10}$$

From (5.7) and (5.10), the total Hamiltonian can be computed:

$$H = \frac{1}{2}\int dk \left\{ \left(m_2^2-m_1^2\right)\left(m_3^2-m_1^2\right)\left(m_4^2-m_1^2\right)\phi_1^*\phi_1 + \left(m_1^2-m_2^2\right)\left(m_3^2-m_2^2\right)\left(m_4^2-m_2^2\right)\phi_2^*\phi_2 \right.$$

$$\left. + \left(m_1^2-m_3^2\right)\left(m_2^2-m_3^2\right)\left(m_4^2-m_3^2\right)\phi_3^*\phi_3 + \left(m_1^2-m_4^2\right)\left(m_2^2-m_4^2\right)\left(m_3^2-m_4^2\right)\phi_4^*\phi_4 \right\} \quad (5.11)$$

Assuming $m_1^2 < m_2^2 < m_3^2 < m_4^2$, we see that the coefficients of $\phi_1^*\phi_1$ and $\phi_3^*\phi_3$ are possitives, while those of $\phi_2^*\phi_2$ and $\phi_4^*\phi_4$ are negative (compare with [13] and Appendix C of reference [3]).

Now that the pattern is clearly established, we can generalize the results to any number of massive states.

The equation of motion is:

$$\prod_{j=1}^{M}\left(\Box + m_i^2\right)\phi = 0 \quad (5.12)$$

Or equivalently

$$\sum_{j=0}^{M} a_j \Box^j \phi \equiv \sum_{j=0}^{M} a_j \phi^{(j)} = 0 \quad (5.13)$$

with

$$a_M = 1 \quad ; \quad a_{M-1} = \sum_{j=1}^{M} m_j^2 \quad ; \quad a_{M-2} = \sum_{j\neq k}{}' \, m_j^2 m_k^2$$

$$\ldots . a_{M-S} = \sum{}' \, m_{j_1}^2 \ldots m_{j_s}^2 , \ldots , a_o = \prod_{j=1}^{M} m_j^2 \quad (5.14)$$

In (5.14) the symbol $\sum'$ means that all the indices are different.

The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\,\phi \prod_{j=1}^{M}\left(\Box+m_i^2\right)\phi \equiv \frac{1}{2}\,\phi \sum_{j=0}^{M} a_j\phi^{(j)} \qquad (5.15)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = a_o\phi + \frac{1}{2}\sum_{j=1}^{M} a_j\phi^{(j)} \quad ; \quad \frac{\partial\mathcal{L}}{\partial\phi^{(j)}} = \frac{1}{2}\, a_j\phi \quad (j\neq 0) \qquad (5.16)$$

From (3.5) and (5.15), (5.16) we get

$$\mathcal{H} = \frac{1}{2}\sum_{st} a_{s+t+1}\left(\phi^{(s)}\ddot{\phi}^{(t)} - \dot{\phi}^{(s)}\dot{\phi}^{(t)}\right) - \mathcal{L} \qquad (5.17)$$

The Fourier development of $\phi(x)$ leads to the representation:

$$\phi(x) = \int dk\, e^{ikr}\sum_{j=1}^{M}\frac{1}{2\omega_j}\left(\phi_j(k)e^{-i\omega_j t} + \phi_j^*(-k)e^{i\omega_j t}\right) \qquad (5.18)$$

where

$$\omega_j = +\sqrt{k^2+m_j^2}$$

Of course

$$\phi^{(\ell)}(x) = \Box^{\ell}\phi(x) =$$

$$\int dk\, e^{ikr}\sum_{j=I}^{M}\frac{\left(-m_j^2\right)^{\ell}}{2\omega_j}\left(\phi_j(k)e^{-i\omega_j} + \phi_j^*(-k)e^{i\omega_j t}\right) \qquad (5.19)$$

From which, (5.17) takes the form:

$$H = \frac{1}{2}\int dk \sum_j \prod_{\ell\neq j} (m_\ell^2-m_j^2)\phi_j^*\phi_j \tag{5.20}$$

Taking $m_1^2 < m^2 < \ldots < m_M^2$, the sign of the coefficient of $\phi_j^*\phi_j$ is seen to be $(-1)^{j+1}$.

Heisenberg equation (4.21) implies

$$\left[H,\phi_j(\vec{k})\right] = -\ \omega_j\phi_j(\vec{k}) \tag{5.21}$$

Which, together with (5.20) gives:

$$\left[\phi_j(\vec{k}),\phi_j^*(\vec{k})\right] = \frac{2\omega_j}{C_j^2}\, S_j\delta(\vec{k}-\vec{k}') \tag{5.22}$$

where

$$c_j^2 = \left|\prod_{\ell\neq j} \left(m_\ell^2-m_j^2\right)\right|$$

and

$$S_j = \text{sign of} \prod_{\ell\neq j}\left(m_\ell^2-m_j^2\right) \tag{5.23}$$

Redefining now the operators ϕ_j:

$$\frac{C_j}{\sqrt{2\omega_j}}\,\phi_j(\vec{k}) \longrightarrow \phi_j(\vec{k}) \tag{5.24}$$

We finally obtain the commutation relations:

$$\left[\phi_j(\vec{k}),\phi_\ell^*(\vec{k}')\right] = S_j\delta_{j\ell}\delta(\vec{k}-\vec{k}') \tag{5.25}$$

and the hamiltonian:

$$H = \int dk \sum_j S_j \phi_j^* \phi_j$$

$$H = \int dk \sum_j (-1)^{j+1} \phi_j^* \phi_j \qquad (5.26)$$

6 PROPAGATORS

Looking at (5.12), we see that the propagator should, at least formally, be given by:

$$\Delta(x) = F\left\{\frac{1}{\prod_{j=1}^{M} (-p^2+m_j^2)}\right\} \qquad (6.1)$$

Where F means Fourier transform.

Of course, the right hand side of (6.1) is not well defined, as it has poles at each m_i^2, so a prescription must be given, for the effective calculation of F. This prescription is equivalent to the choice of boundary conditions. The simplest way to express quantum causality is by analytic continuation in the coefficients of the metric [14], going from euclidean to hyperbolic one.

Explicitly, one starts from an euclidean space:

$$p^2 = p_1^2 + p_2^2 + p_3^2 + a^2 p_4^2$$

In this way we get analytic functions (distributions) of a, which are continued to $a = i + \varepsilon$, generating the well known "$i\varepsilon$" (See reference [15]).

Formula (6.1) will be well defined by the prescription $p^2 \to p^2 + i\varepsilon$.

Alternatively we can take the usual definition:

$$\Delta_c(x) = \langle 0 | T\phi(x)\phi(0) | 0 \rangle \tag{6.2}$$

Defining now the partial fields:

$$\phi_j(x) = \int dk \frac{e^{ikr}}{2\omega_j} \left(\phi_j(k) e^{-i\omega_j t} + \phi_j^*(-k) e^{i\omega_j t} \right) \tag{6.3}$$

with

$$\phi(x) = \sum_{j=1}^{M} \phi_j(x) \tag{6.4}$$

And using the commutation relations (5.22) we arrive at :

$$\Delta_c(x) = \sum_j \langle 0 | T\phi_j(x)\phi_j(0) | 0 \rangle$$

$$\Delta_c(x) = \sum_{j=1} \frac{S_j}{C_j^2} \Delta_F^{(j)}(x) \tag{6.5}$$

Where $\Delta_F^{(j)}(x)$ is the usual Feynman propagator for a particle with mass m_j.

Taking the Fourier transform of (6.5), we get:

$$\Delta_c(p) = \sum_{j=1}^{M} \frac{S_j}{C_j^2} \frac{1}{m_j^2 - p^2 - io} \tag{6.6}$$

Or, taking into account (5.23):

$$\Delta_c(p) = \sum_{j=1} \frac{1}{\prod_{\ell \neq j} \left(m_\ell^2 - m_j^2\right)} \frac{1}{\left(m_j^2 - p^2 - i\sigma\right)} \tag{6.7}$$

Which is nothing but

$$\Delta_c(p) = \frac{1}{\prod_{j=1}^{M} \left(m_j^2 - p^2 - io\right)} \tag{6.8}$$

In coincidence with (6.1) except taht the prescription "io" is incorporated in (6.8).

Note that the rule $p^2 \to p^2 + io$, is equivalent to the usual one $m^2 \to m^2 - io$, for propagators of the form (6.1). However this is not true in general. Take for example equation (4.7) whose formal Green function is:

$$\Delta = F\left\{\frac{1}{p^4 - \mu^4}\right\} = F = \left\{\frac{1}{p^2 - \mu^2} \frac{1}{p^2 + \mu^2}\right\}$$

$$\Delta = \frac{1}{2\mu^2} F\left\{\frac{1}{p^2 - \mu^2}\right\} - \frac{1}{2\mu^2} F\left\{\frac{1}{p^2 + \mu^2}\right\} \tag{6.9}$$

It is evident that for the second Fourier transform in (6.9) the prescriptions do not coincide.

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