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A COVARIANT WAVE EQUATION IN THE QUANTUM GEOMETRY OF BOSONIC STRINGS

by

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Abstract

In this letter we propose a covariant wave equation in the Polyakov's Quantum Geometry of Bosonic Strings. Its main feature are the explicit appearence of a metric dependent term (vanishing at D=26) due to the conformal anomaly and its reduction to the Nambu String Wave Equation for D=26, where D is the space-time dimension.

1. Introduction

Recently, A.A.Tseytlin ([1]) suggested a new approach to the covariant formulation of the second-quantized Bose string field theory based on the Polyakov's quantum geometry.

A basic problem in this second-quantized formulation concerns the determination of the wave equation satisfied by the string wave functional which affords the construction of the string Feynmann rules and study of the theory ground states.

Our aim in this letter is to propose a covariant wave equation for the Polyakov bosonic string (in the cylindrical conformal gauge) by taking into account in a explicitly analysis the presence of the conformal anomaly of the theory.

2. The Wave Equation

The basic object in a covariant second quantized string theory is the quantum string state which is represented by a functional $\Phi[\mathcal{C}] = \Phi[X^\mu(\sigma), e(\sigma)]$ of a parametrized contour $X^\mu(\sigma)$ $(0 \le \sigma \le 1)$ and of a one dimensional intrinsic metric $e(\sigma)$. The quantum free propagation of a initial string state $\Phi[\mathcal{C}^{in}]$ for a final state $\Phi[\mathcal{C}^{out}]$ is furnished by the evolution equation ([1]; [6]).

$$\Phi[\mathcal{C}^{out}] = \int [d\mathcal{C}^{in}]\, G[\mathcal{C}^{out}; \mathcal{C}^{in}]\, \Phi[\mathcal{C}^{in}] \qquad (1)$$

where the string propagation Kernel $G\,[C^{OUT};\ C^{IN}]$ can be expressed as a Polyakov surface sum over cylindrical 2-surface with zero Euler number, that is

$$G[C^{OUT};C^{IN}] = \int_{\partial M = (C^{OUT};\,C^{IN})} \mathcal{D}[M]\ \mathrm{EXP}\left(-I_0\left(g_{ab}(\sigma,\tau),X_\mu(\sigma,\tau)\right)\right) \tag{2}$$

$$\mathcal{D}[M] = \left[d\,g_{ab}(\sigma,\tau)\ d X_\mu(\sigma,\tau)\right]$$

The metric $\{g_{ab}(\sigma,\tau)\}$ on M can be choosen to satisfy the so called conformal cylindrical gauge

$$\begin{aligned} g_{ab}(\sigma,\tau) &= \exp\ \varphi(\sigma,\tau)\,\delta_{ab} \\ &= \rho(\sigma,\tau)\ \delta_{ab} \end{aligned} \tag{3.A}$$

$$0 \leq \tau \leq T\ ;\ 0 \leq \sigma \leq 1$$

so that

$$C^{IN} = \left\{X^\mu_{IN}(\sigma) = X^\mu(\sigma,0);\ e_{IN}(\sigma) = e^{\varphi(\sigma,0)}\right\} \tag{3.B}$$

$$C^{OUT} = \left\{X^\mu_{OUT}(\sigma) = X^\mu(\sigma,T);\ e_{OUT}(\sigma) = e^{\varphi(\sigma,T)}\right\} \tag{3.C}$$

The path measure $[d\,C^{IN}]$ is defined by the covariant expression.

$$[d\ell^{iN}] = \left(\prod_{0\le\sigma\le 1} de_{iN}(\sigma)\cdot dX^{\mu}_{iN}(\sigma)\right)$$
$$\times\ \exp\left(-\int_0^1 |X'_{iN}(\sigma)|^2/e_{iN}(\sigma) + \frac{1}{2}e_{iN}(\sigma)\right) \quad (4)$$

We note that the boundary term associated to the initial string state C^{IN} in the Brink-Di Vecchia-Howe string covariant action can be naturally absorbed in eq.(4). Procedure assumed in the discussion that follows.

Let us now present our basic idea to write a covariant string wave equation for the string state $\Phi[C^{IN}]$ possessing as "string green function" the quantum surface path integral given by eq. (2).

In order to write a wave equation associated to the "Kernel" eq.(2), we consider variations on "time direction" $\mathcal{T}$ of the C^{IN} end contour in the above quoted surface path integral.

It is worth to remark that this procedure to deduce a wave equation for the "kernel" eq.(2) is similar to that used to write the Wheeler-De Witt equation in the path integral formulation in Quantum Gravity (see Ref.[2] - Sec.B).

Since $\mathcal{T}$-variations on string surface are equivalent to $g_{00}(\sigma,\mathcal{T})$ variations and these variations should vanish we arrive at the following identity:

$$0 = \lim_{\tau \to 0^+} \left\langle T_{00}\left(X^\mu(\sigma,\tau), \dot{X}_\mu(\sigma,\tau), X'^\mu(\sigma,\tau), g_{ab}(\sigma,\tau) \right) \right\rangle$$
$$+ \lim_{\tau \to 0^+} \left\langle \frac{\delta}{\delta g_{00}(\sigma,\tau)} \exp\left[-F(g_{ab}(\sigma,\tau)) \right] \right\rangle \quad (5)$$

where $\langle \quad \rangle$ is the Polyakov quantum surface average defined by eq.(2), $T_{00}(X^\mu(\sigma,\tau),\ldots) = \left(\partial_\tau X^\mu \cdot \partial_\tau X^\mu - \frac{1}{2} g^{00} g_{cd} \times \partial_c X^\mu \cdot \partial_d X^\mu \right)(\sigma,\tau)$ is the time component of the (intrinsic) string energy-momentum tensor. The presence of the term $F[g_{ab}(\sigma,\tau)]$ is due to the conformal anomaly of the theory ([3], [4]). In vanishes for D=26 and its general covariant expression is given by ([3], [4])

$$F[g_{ab}(\sigma,\tau)] = \frac{26-D}{48\pi} \left\{ \int_0^1 d\sigma\, d\sigma' \int_0^T d\tau\, d\tau' \left[(\sqrt{g}\, R)(\sigma,\tau) \right.\right.$$
$$K\left((\sigma,\tau); (\sigma',\tau'), g_{ab}(\sigma',\tau') \right) \quad (6)$$
$$\left.\left. (\sqrt{g}\, R)(\sigma',\tau') \right] + \mu_0^2 \int_0^1 d\sigma \int_0^T d\tau \sqrt{g}(\sigma,\tau) \right\}$$

where $K\left((\sigma,\tau); (\sigma',\tau'), g_{ab}(\sigma',\tau') \right)$ denotes the Green function of the Laplace Beltrami operator associated to the metric $g_{ab}(\sigma',\tau')$ ([3]).

By writing the relationship eq.(5) in the (cylindrical) conformal gauge eq.(3-A) and using the result

$$\frac{\delta}{\delta g_{00}(\xi'')} K\left(\xi;\xi', g_{ab}(\xi'_0,\xi'_1) = e^{\varphi(\xi'_0,\xi'_1)}\delta_{ab}\right) = \frac{1}{2\pi}\delta^{(2)}(\xi-\xi'')$$

([3]), we obtain the following result

$$\Big\langle (D-26)\left(\lim_{\tau\to 0^+}\frac{1}{24\pi^2}R\left(e^{\varphi(\sigma,\tau)}\right)+\frac{\mu_0^2}{24\pi}\right)$$

$$+\left(\Pi^{\mu}_{iN}(\sigma)^2-\frac{1}{2}|X'_{\mu}(\sigma)|^2\right)\Big\rangle = 0 \qquad (7)$$

Here, $\Pi^{\mu}_{iN}(\sigma) = \lim_{\tau\to 0^+}\partial_\tau X^{\mu}(\sigma,\tau)$ denotes the canonical momentum associated to the string vector position $X^{iN}_{\mu}(\sigma)$ and $R\left(e^{\varphi(\sigma,\tau)}\right) = -e^{-\varphi(\sigma,\tau)}\left((\partial_\tau\varphi)^2+(\partial_\sigma\varphi)^2\right)(\sigma,\tau)$ is the scalar of curvature associated to the metric $g_{ab}(\sigma,\tau) = e^{\varphi(\sigma,\tau)}\delta_{ab}$

At this point we propose to enforce eq.(7) as an operator identity on the state space of strings functionals $\Phi[C]$.

In the operator quantization framework we have to impose covariant commutation relations for the dynamical degrees of freedom of the dynamical system under study.

For the Polyakov string the dynamical degrees of freedom are the vector position $X^{\mu}_{iN}(\sigma)$ and the intrinsic metric $e_{iN}(\sigma)$. So, in order to write covariant commutation relations one should consider, besides the usual canonical string vector position momentum $\Pi^{\mu}_{iN}(\sigma)$; a canonical momentum associated to the $e_{iN}(\sigma)$ variable. A natural candidate for this object is given by the expression $\underset{\sim}{\Pi}^{iN}(\sigma) = \lim_{\tau\to 0^+}\partial_\tau\varphi(\sigma,\tau)$ ([4]).

The covariant commutation relations satisfied by these dynamical variables are

$$\left[\Pi^{\mu}_{iN}(\sigma) , X^{\nu}_{iN}(\sigma') \right] = \delta^{\mu\nu} \frac{1}{e(\sigma')} \delta(\sigma - \sigma')$$

$$\left[\underset{\sim}{\Pi}_{iN}(\sigma) , e(\sigma') \right] = \frac{1}{e(\sigma')} \delta(\sigma - \sigma') \qquad (8)$$

In the Schroedinger representation it is easily verified the following explicit representation for the canonical momenta $\left\{ \Pi^{\mu}_{iN}(\sigma) , \underset{\sim}{\Pi}_{iN}(\sigma) \right\}$ ([5]) :

$$\Pi^{\mu}_{iN}(\sigma) = \frac{1}{e_{iN}(\sigma)} \frac{\delta}{\delta X^{iN}_{\mu}(\sigma)} \qquad (9\text{-A})$$

$$\underset{\sim}{\Pi}_{iN}(\sigma) = \frac{1}{e_{iN}(\sigma)} \frac{\delta}{\delta e_{iN}(\sigma)} \qquad (9\text{-B})$$

Thus, using eq.(9-A) - eq.(9-B) in the relation eq.(7) we translate the classical identity eq.(7) into the following operator identity:

$$\left\{ \frac{26-D}{24\pi^2} \frac{1}{e_{iN}(\sigma)} \left(\frac{1}{e_{iN}(\sigma)^2} \frac{\delta^2}{\delta e_{iN}(\sigma)\,.\delta e_{iN}(\sigma)} + (e'_{iN}(\sigma))^2 + \mu_0^2 \pi \; e_{iN}(\sigma) \right) + \left(\frac{1}{e^2_{iN}(\sigma)} \frac{\delta^2}{\delta X^{iN}_{\mu}(\sigma)\,.\delta X^{iN}_{\mu}(\sigma)} - \frac{1}{2} |X'_{iN}(\sigma)|^2 \right) \right\} \Phi[X^{iN}_{\mu}(\sigma), e^{iN}(\sigma)] = 0 \tag{10}$$

The operator equation writen above is our proposed covariant wave equation in the quantum geometry of bosonic strings.

By analysing eq.(10) we can see in a invariant way the reduction at D=26 to the Nambu string wave equation associated to the Veneziano dual bosonic model [6] by choosing the string "proper-time" gauge $e_{iN}(\sigma)$ = constant.

Extension of this study to the important fermionic case ([7], [8]) will be discussed in other paper.

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