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Z(4) MODEL: CRITICALITY AND BREAK-COLLAPSE METHOD

by

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ABSTRACT

Within a real-space renormalization group (RG) framework, we study the criticality of the Z(4) ferromagnet on square lattice. The phase diagram (exhibiting ferromagnetic, paramagnetic and nematic-like phases) recovers *all* the available *exact* results, and possibly is a high precision one everywhere. In particular we establish the main asymptotic behaviors (bifurcation and Ising regions). In addition to that, we develop an operational procedure (Break-collapse method) which considerably simplifies the *exact* calculation of arbitrary Z(4) two-terminal clusters (commonly appearing in RG approaches).

Key-words: Z(4) model; Phase diagram; Renormalization group; Square lattice.

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The Z(N) model contains, as particular cases, various important statistical models (e.g., bond percolation, spin 1/2 Ising and Potts models) and is relevant for a large class of physical problems (e.g., random resistor and magnetic systems, adsorption). It has attracted, during last years, a certain amount of work[1-8], mainly in two dimensions, and addressing more particularly the square lattice which, due to its self-duality, turns out to be relatively simple. The Z(N) model starts being larger than the N-state Potts model at N=4. The phase diagram of the Z(4) ferromagnet in square lattice is known to present three phases, namely the paramagnetic (P; Z(4) symmetry), the nematic-like or intermediate (I; Z(2) symmetry) and the ferromagnetic (F; completely broken symmetry) ones. The entire phase diagram is constituted by second or higher order phase transitions. The P-F, I-F and I-P critical lines join at the 4-state Potts critical point. The P-F line is entirely determined by duality arguments; furthermore these arguments biunivocally relate the *still unknown* (as far as we know) I-F and I-P lines. Herein we calculate these lines by constructing a real-space renormalization-group (RG) based in the well known self-dual Wheatstone bridge cluster (Fig. 1).

A convenient form for the Z(4) (symmetric Ashkin-Teller model) ferromagnet (dimensionless) Hamiltonian is the following[7]:

$$\frac{\mathcal{H}}{k_BT} = \sum_{<i,j>} [K_1 - K_1(\sigma_i\sigma_j + \tau_i\tau_j) - 2K_2(\sigma_i\sigma_j\tau_i\tau_j)] \tag{1}$$

where T is the temperature, <i,j> runs over all first-neighboring pairs of sites on a square lattice, $\sigma_i = \pm 1$, $\tau_i = \pm 1$ $(\forall i)$, $K_1 \geq 0$ and $K_1 + 2K_2 \geq 0$. Let us introduce the operationally convenient var iable (*vector transmissivity*[7]) $\vec{t} \equiv (1, t_1, t_2, t_3)$ through

$$t_1 = t_3 = \frac{1-e^{-4K_1}}{1+2\ e^{-2(K_1+2K_2)} + e^{-4K_1}} \tag{2.a}$$

$$t_2 = \frac{1-2\ e^{-2(K_1+2K_2)} + e^{-4K_1}}{1+2\ e^{-2(K_1+2K_2)} + e^{-4K_1}} \tag{2.b}$$

This vector transmissivity generalizes the scalar one used in the Ising (recovered as $t_2=t_1^2$) and in the 4-state Potts (recovered as $t_1=t_2$) models[9]. The transmissivity $\vec{t}^{(s)}$ ($\vec{t}^{(p)}$) corresponding to a series (parallel) array of two bonds, respectively associated with $\vec{t}^{(1)}$ and $\vec{t}^{(2)}$, is given by[7]

$$t_i^{(s)} = t_i^{(1)}\, t_i^{(2)} \quad (i=1,2) \quad \text{(series)} \tag{3}$$

and

$$t_i^{(p)D} = t_i^{(1)D}\, t_i^{(2)D} \quad (i=1,2) \quad \text{(parallel)} \tag{4}$$

where the *dual* transmissivity $\vec{t}^D$ is given by

$$t_1^D \equiv \frac{1-t_2}{1+2t_1+t_2} \tag{5.a}$$

$$t_2^D \equiv \frac{1-2t_1+t_2}{1+2t_1+t_2} \tag{5.b}$$

Algorithms (3) and (4) enable quick calculation of the transmissivity corresponding to *any* two-terminal array fully reducible in series/parallel sequences; we shall see later on how to deal with arrays which are not fully reducible.

To treat Hamiltonian (1) we use the cluster RG transformation indicated in Fig. 1 (with $\vec{\vec{t}}=\vec{t}$). This choice is known to be a very convenient one for the square lattice (e.g., random resistor [10], bond percolation [11], N-state Potts [9], anisotropic Heisenberg [12] models). The RG recursive relations are constructed to preserve the correlation function, i.e., $\exp(-\mathcal{H}'_{12}/k_BT) = \mathrm{Tr}_{3,4}\exp(-\mathcal{H}_{1234}/k_BT)$, where $\mathcal{H}'_{12}$ and $\mathcal{H}_{1234}$ are the Hamiltonians respectively associated with Figs. 1(a) and 1(b). ($\mathcal{H}'_{12}$ includes an additive constant). We obtain, through a tedious but straightforward calculation,

$$t'_1 = \frac{[2(1+t_2^2)t_1^2] + [2(1+t_2)^2t_1^2]\bar{t}_1 + [4t_2t_1^2]\bar{t}_2}{[1+t_2^4+2t_1^4] + [4(1+t_2^2)t_1^2]\bar{t}_1 + [2(t_2^2+t_1^4)]\bar{t}_2} \tag{6}$$

$$t'_2 = \frac{[2(t_2^2+t_1^4)] + [8t_2t_1^2]\bar{t}_1 + [2(t_2^2+t_1^4)]\bar{t}_2}{[1+t_2^4+2t_1^4]+[4(1+t_2^2)t_1^2]\bar{t}_1+[2(t_2^2+t_1^4)]\bar{t}_2} \tag{7}$$

where t_1 and t_2 (t'_1 and t'_2) are related to K_1 and K_2 (K'_1 and K'_2) through Eqs. (2), and where $\vec{\vec{t}}=\vec{t}$ (it is only for future convenience that we have already indicated the result corresponding to Fig. 1(b), where $\vec{t}$ and $\vec{\vec{t}}$ are arbitrary transmissivities). Eqs.(6) and (7) fully determine the phase diagram we are looking for, as well as thermal-type critical exponents.

Before analysing the results, let us describe a particularly simple manner (*break-collapse method*; BCM) to obtain Eqs. (6) and (7), and, more generally speaking, the equivalent transmissivity $\vec{G}$ associated with an *arbitrary* (series/parallel reducible or not, planar or not) two-terminal graph of $Z(4)$ bonds. $\vec{G}$ is determined by $G_\ell(\{\vec{t}^{(i)}\}) = N_\ell(\{\vec{t}^{(i)}\})/D(\{\vec{t}^{(i)}\})$ ($\ell=1,2$) where $\{\vec{t}^{(i)}\}$ denotes the set of transmissivities respectively associated with the bonds of the graph, and $N_\ell(\{\vec{t}^{(i)}\})$ and $D(\{\vec{t}^{(i)}\})$ are *multilinear* polynomials of the form $A+B\,t_1^{(j)}+C\,t_2^{(j)}$ for an *arbitrary* j^{th}-bond, A, B and C depending on the set of transmissivities (noted $\{\vec{t}^{(i)}\}'$) of the remaining bonds. The performance of three different operations on the j-th bond, namely the "break" ($t_1^{(j)}=t_2^{(j)}=0$), the "collapse" ($t_1^{(j)}=t_2^{(j)}=1$) and the "pre-collapse" ($t_1^{(j)}=0, t_2^{(j)}=1$), completely determine A, B and C. It immediately follows:

$$N_\ell(\{\vec{t}^{(i)}\}) = (1-t_2^{(j)})\,N_\ell^{bb}(\{\vec{t}^{(i)}\}') + t_1^{(j)} N_\ell^{cc}(\{\vec{t}^{(i)}\}') + (t_2^{(j)}-t_1^{(j)})\,N_\ell^{bc}(\{\vec{t}^{(i)}\}') \quad (\ell=1,2) \qquad (8.a)$$

$$D(\{\vec{t}^{(i)}\}) = (1-t_2^{(j)})\,D^{bb}(\{\vec{t}^{(i)}\}') + t_1^{(j)} D^{cc}(\{\vec{t}^{(i)}\}') + (t_2^{(j)}-t_1^{(j)})\,D^{bc}(\{\vec{t}^{(i)}\}') \qquad (8.b)$$

where $N_\ell^{bb},\ldots,D^{bc}$ are the numerators and denominators of the "broken" (bb), "collapsed" (cc) and "pre-collapsed" (bc) graphs. By recursively using this property and algorithms (3) and (4) the problem is easily solved. In other words, *the tracing algebra is automatically satisfied through the (above mentioned) topological operations*. Let us illustrate the procedure on graph of Fig.1(b): its broken, collapsed and pre-collapsed graphs (operating on the $\vec{t}$-bond) are respectively represented in Figs. 2(a), 2(b) and 2(c), and yield

$$N_1^{bb} = 2(1+t_2^2)t_1^2 \qquad (9.a)$$

$$N_2^{bb} = 2(t_2^2+t_1^4) \qquad (9.b)$$

$$D^{bb} = 1 + t_2^4 + 2t_1^4 \tag{9.c}$$

$$N_1^{cc} = 4(1+t_2)^2 t_1^2 \tag{9.d}$$

$$N_2^{cc} = 4(t_2^2+2t_2t_1^2+t_1^4) \tag{9.e}$$

$$D^{cc} = (1+t_2^2)^2+4[(1+t_2^2)t_1^2+t_1^4] \tag{9.f}$$

$$N_1^{bc} = 2(1+t_2)^2 t_1^2 \tag{9.g}$$

$$N_2^{bc} = 4(t_2^2+t_1^4) \tag{9.h}$$

$$D^{bc} = (1+t_2^2)^2 + 4t_1^4 \tag{9.i}$$

Eqs. 9(a)-(f) were obtained through exclusive use of algorithms (3) and (4); Eqs. 9(g)-(i) used also algorithm (8) (and the fact that a graph exclusively made by pre-collapsed bonds is pre-collapsed itself). It can be checked that Eqs. (9) replaced into Eqs. (8) recover Eqs. (6) and (7), *very tedious to establish through the traditional tracing operations*. This type of procedure has been very useful in a variety of problems (Potts[9] model, resistor network[13], directed percolation[14]): it is herein established for the Z(4) model (we are presently working in its generalization for the Z(N) model).

We go now back to the criticality provided by Eqs. (6) and (7) (with $\vec{t}\,'=\vec{t}$). The present RG shares with the Migdal - Kadanoff-like RG of Ref. [6] the fact that it <u>recovers</u> *all* the available *exact* results for the phase diagram of the Z(4) ferromagnet in square lattice (see Fig.3), namely: (i) the self-dual line ($t_2=1-2t_1$), part of which constitutes the P-F critical line; (ii) the location of the Potts ($t_1=t_2=1/3$; P point in Fig. 3), Ising-1 ($t_1=\sqrt{t_2}=\sqrt{2}-1$; I_1 point), Ising-2 ($t_1=0$, $t_2=\sqrt{2}-1$; I_2 point) and Ising-3 ($t_1=\sqrt{2}-1$, $t_2=1$; I_3 point) critical points; (iii) the I-F and I-P critical lines are related through duality (Eqs. (5)); (iv) the phase transitions are 2^{nd} or higher order ones. Furthermore the present RG provides the following asymptotic behaviors (possibly excellent for the square lattice):

$$t_2 \sim (\sqrt{2}-1) - ct_1^3 \quad (c=2(3\sqrt{2}-2)/7 \simeq 0.64) \tag{10}$$

$$t_2 \sim 1-d[(\sqrt{2}-1)-t_1] - e[(\sqrt{2}-1)-t_1]^3 \tag{11}$$

$$(d=2/(\sqrt{2}-1) \simeq 4.83; \quad e=c/\sqrt{2}(\sqrt{2}-1)^4 \simeq 15.4)$$

$$t_2 \sim 1-2t_1 \pm f(1/3-t_1)^{\phi} \quad (f \simeq 982; \phi = \ln(27/13)/\ln(17/13) \simeq 2.7245) \tag{12}$$

in the neighborhood of the I_2, I_3 and P points respectively. With respect to the thermal critical exponent ν, the results are the following: (i) at all three Ising points, $\nu=\ln 2/\ln(2\sqrt{2}-1) \simeq 1.149$ (ν(exact)=1); (ii) at the Potts point, $\nu=\ln 2/\ln(27/13) \simeq 0.948$ (ν exact)=2/3); (iii) the I-F and I-P lines belong to the Ising universality class (which is known to be correct); (iv) the P-F line belong to the Ising universality class (which is wrong[15]; this error could possibly disappear in the increasingly large cluster limit).

Summarizing, the Z(4) ferromagnet phase diagram obtained within the present RG approach recovers *all* (as far as we know) the *exact* results available for the square lattice, and possibly is an excellent approximation everywhere (in particular, the asymptotic behaviors (10)-(12)); the approach is less performant for the thermal critical exponents. If instead of the square lattice, we focuse the hierarchical one generated by transformation in Fig. 1, then *all* the present results are exact. In addition to that, we have established a new method for calculating *arbitrary* two-terminal (and possibly n-terminal) arrays of Z(4) (and possibly Z(N) within appropriate generalization) bonds. The procedure is operationally quite convenient as the tedious tracing algebraic calculations are automatically performed through elementary topological operations. Consequently RG's based in relatively large clusters become tractable.

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FIGURE CAPTIONS

Fig. 1 - RG clusters. o(●) denotes terminal (internal) node.

Fig. 2 - Broken ((a)), collapsed ((b)) and pre-collapsed ((c)) graphs, obtained from that of Fig. 1(b), considering respectively $\bar{t}_1=\bar{t}_2=0$; $\bar{t}_1=\bar{t}_2=1$ and $\bar{t}_1=0$, $\bar{t}_2=1$.

Fig. 3 - (a): Phase diagram in the (t_1,t_2) space. F, I, and P respectively, denote the ferromagnetic, intermediate and paramagnetic phases. P is the Potts fixed point; I_1, I_2 and I_3 are the Ising fixed points. ■ denotes the fully stable fixed points. The shaded region is non physical. The $t_2=t_1$ and $t_2=t_1^2$ dashed lines respectively represent Potts and Ising invariant subspaces.

b) Phase diagram in the $(k_BT/J_1, 1+2J_2/J_1)$ space. $J_i=$ $=k_BT\ K_i$ i=1,2). The dashed lines are asymptots.

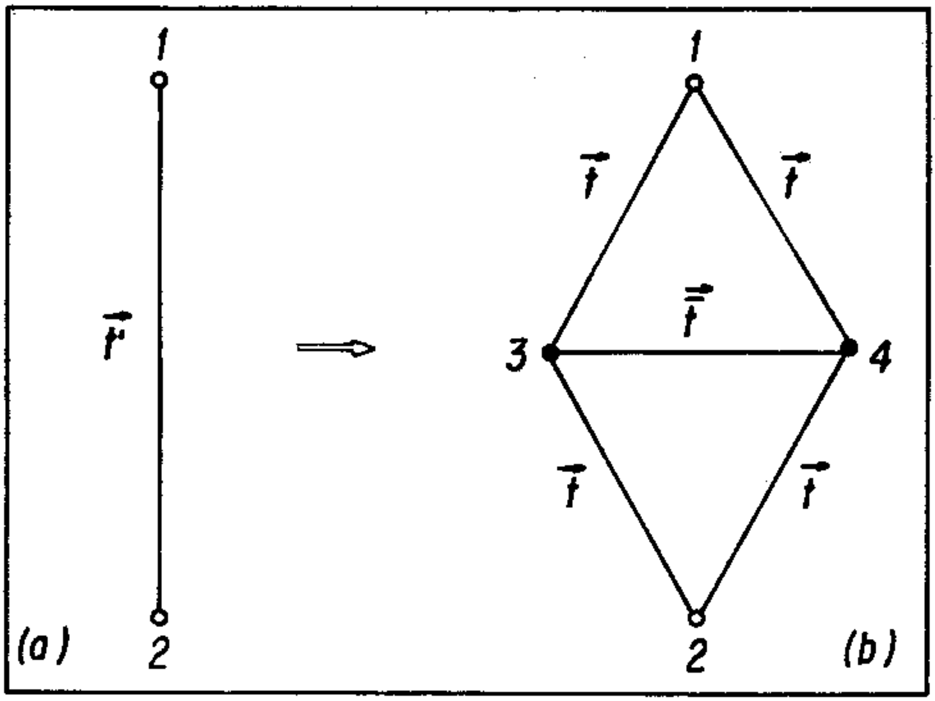

FIG.1

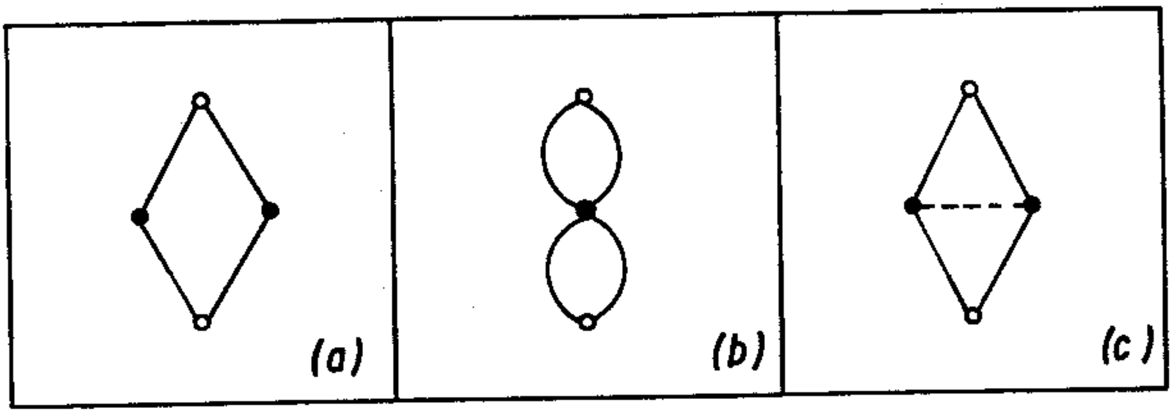

FIG.2

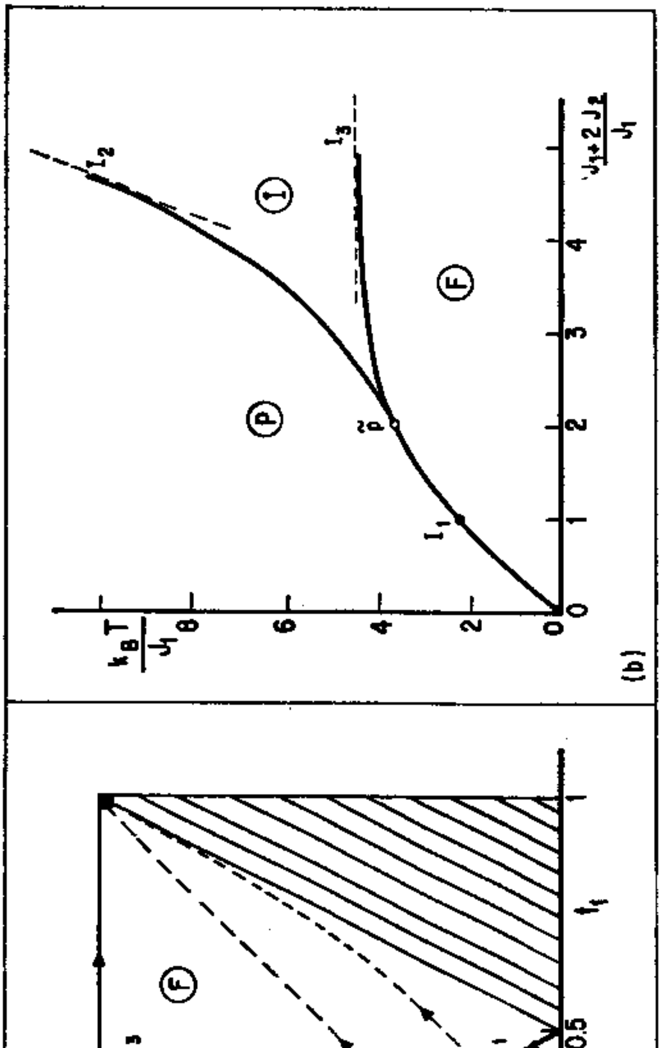

FIG.3

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