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# CONDUCTIVITY OF A SQUARE-LATTICE BOND-MIXED RESISTOR NETWORK

by

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ABSTRACT

Within a real-space renormalization-group framework based on self-dual clusters, we calculate the conductivity of a square-lattice quenched bond-random resistor network, the conductance on each bond being $g_1$ or $g_2$ with probabilities (1-p) and p respectively. The group recovers several already known exact results (including slopes), and is consequently believed to be numerically quite reliable for almost all values of p, and all ratios $g_1/g_2$ (in particular, $g_1=0$ and $g_1=\infty$ with finite $g_2$ respectively correspond to the insulator-resistor and superconductor-resistor mixtures). In addition to that, we propose an heuristic *analytic* expression for the conductivity which is believed to be a quite satisfactory approximation everywhere not too close to the percolation point.

Key-words: Resistor network; Conductivity; Renormalization group; Bond-random square lattice.

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# I INTRODUCTION

Electrical conduction in random resistor networks and the associated criticality have been the theme of a considerable amount of efforts during recent years. Theoretical approaches such as computational simulations[1,2], renormalization groups[3,4] and others[5-11], as well as experimental results[12-14] are already available. Nevertheless the problem is far from being fully solved, even for very simple systems such as the square lattice with quenched binary distribution of conductances (bond conductance $g_1$ or $g_2$ with probabilities (1-p) and p respectively). The corresponding *exact* functional dependence of the conductivity $\sigma$ on p, $g_1$ and $g_2$ is still unknown.

In the present paper we introduce a real-space renormalization-group (RG) formalism which follows along the lines of those recently developed in Refs. [15,16] to treat the conductivity of simpler but related systems. In Section II we introduce the model and the RG formalism, in Section III we present the results, and finally we conclude in Section IV.

# II MODEL AND RENORMALIZATION GROUP

We consider a square lattice with the following conductance distribution associated with each bond:

$$P(g) = (1-p)\delta(g-g_1) + p\delta(g-g_2) \qquad (g_1, g_2 \geq 0) \tag{1}$$

The conductance of a parallel or series array of two bonds (with conductances $\bar{g}_1$ and $\bar{g}_2$) is respectively given by

$$g_p = \bar{g}_1 + \bar{g}_2 \qquad \text{(parallel)} \tag{2}$$

$$g_s = \bar{g}_1\bar{g}_2/(\bar{g}_1+\bar{g}_2) \qquad \text{(series)} \tag{3}$$

The latter can be written in the same form as the former, namely

$$g_s^D = \bar{g}_1^D + \bar{g}_2^D \tag{4}$$

with

$$g^D \equiv g_0^2/g \qquad (i=1,2,s) \tag{5}$$

where D stands for *dual*[15] (see also Refs. [17-19] for a related discussion in the context of the Potts and Z(N) models), and $g_0$ is an arbitrary reference conductance.

Let us now introduce the following convenient variable[15]

$$S \equiv \frac{g}{g+g_0} \tag{6}$$

which satisfies an interesting (probability like) property, namely

$$S^D(g) \equiv S(g^D) = 1-S(g) \tag{7}$$

where we have used definition (5). On the basis of this S-variable it will be possible later on to construct a quite performant RG (similarly to what occurred for the bond-dilute problem[15]).

We next introduce the RG formalism which yields $\sigma(g_1,g_2,p)$, by renormalizing the self-dual Wheatstone bridge cluster (Fig. 1(b)) into a single bond (Fig. 1(a)) (the RG linear scale factor b equals 2). The conductance $g_H$ of a Wheatstone bridge with elementary conductances $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_5$ as indicated in Fig. 1(b) is given (see for instance [15]) by

$$g_H = \frac{\bar{g}_1\bar{g}_2\bar{g}_3 + \bar{g}_1\bar{g}_2\bar{g}_4 + \bar{g}_2\bar{g}_3\bar{g}_4 + \bar{g}_1\bar{g}_3\bar{g}_4 + \bar{g}_5(\bar{g}_1\bar{g}_3 + \bar{g}_2\bar{g}_3 + \bar{g}_1\bar{g}_4 + \bar{g}_2\bar{g}_4)}{\bar{g}_1\bar{g}_2 + \bar{g}_1\bar{g}_3 + \bar{g}_2\bar{g}_4 + \bar{g}_3\bar{g}_4 + \bar{g}_5(\bar{g}_1 + \bar{g}_2 + \bar{g}_3 + \bar{g}_4)} \qquad (8)$$

Consequently the distribution law $P_H$ associated with Fig. 1(b) if each one of its bonds is associated with the distribution $P(g)$ (Eq. (1)) is given by

$$\begin{aligned}
P_H(g) = {} & [(1-p)^5 + (1-p)^4 p]\, \delta(g-g_1) + 4(1-p)^4 p\, \delta\Big(g - \frac{3g_1^2 + 5g_1g_2}{5g_1 + 3g_2}\Big) \\
& + 2(1-p)^3p^2\, \delta\Big(g - \frac{g_1^2 + 4g_1g_2 + 3g_2^2}{2g_1 + 6g_2}\Big) + 2(1-p)^3p^2\, \delta\Big(g - \frac{2g_1g_2}{g_1+g_2}\Big) \\
& + 4(1-p)^3p^2\, \delta\Big(g - \frac{g_1^3 + 5g_1^2g_2 + 2g_1g_2^2}{2g_1^2 + 5g_1g_2 + g_2^2}\Big) + 2(1-p)^3p^2\, \delta\Big(g - \frac{g_1^3 + 4g_1^2g_2 + 3g_1g_2^2}{3g_1^2 + 4g_1g_2 + g_2^2}\Big) \\
& + 2(1-p)^2p^3\, \delta\Big(g - \frac{g_2^3 + 4g_2^2g_1 + 3g_2g_1^2}{3g_2^2 + 4g_2g_1 + g_1^2}\Big) + 4(1-p)^2p^3\, \delta\Big(g - \frac{g_2^3 + 5g_2^2g_1 + 2g_2g_1^2}{2g_2^2 + 5g_2g_1 + g_1^2}\Big) \\
& + 2(1-p)^2p^3\, \delta\Big(g - \frac{2g_2g_1}{g_2+g_1}\Big) + 2(1-p)^2p^3\, \delta\Big(g - \frac{g_2^2 + 4g_2g_1 + 3g_1^2}{2g_2 + 6g_1}\Big) \\
& + 4(1-p)p^4\, \delta\Big(g - \frac{3g_2^2 + 5g_2g_1}{5g_2 + 3g_1}\Big) + [(1-p)p^4 + p^5]\, \delta(g-g_2) \qquad (9)
\end{aligned}$$

We could in principle follow the evolution, under successive renormalizations, of the distribution law until it attains an invariant form. This procedure has in fact already been used[3] for random resistor problems. However an operationally much simpler and numerically excellent procedure (which has yielded quite satisfactory results for the Potts model[19]) can be followed instead, namely to approximate distribution $P_H(g)$ by a binary one, given by

$$P'(g) = (1-p')\delta(g-g_1') + p'\delta(g-g_2') \tag{10}$$

where $p'$, $g_1'$ and $g_2'$ will be completely determined (as functions of $(p,g_1,g_2)$) by imposing the invariance of the three first momenta of a function $f(g)$ to be chosen. A natural possible choice is $f(g) = g$, and we shall denote g-RG the corresponding RG. However, a more sophisticated and convenient choice is possible[15] namely $f(g) = S(g)$ (we denote S-RG the corresponding RG). More precisely, we impose

$$\langle S(g)\rangle_{P'} = \langle S(g)\rangle_{P_H} \tag{11.a}$$

$$\langle [S(g)]^2\rangle_{P'} = \langle [S(g)]^2\rangle_{P_H} \tag{11.b}$$

$$\langle [S(g)]^3\rangle_{P'} = \langle [S(g)]^3\rangle_{P_H} \tag{11.c}$$

hence

$$(1-p')S_1' + p'S_2' = [(1-p)^5 + (1-p)^4 p]S_1 + 4(1-p)^4 p\, S\left(\frac{3g_1^2 + 5g_1g_2}{5g_1 + 3g_2}\right) + \ldots \equiv F(p,S_1,S_2) \tag{12.a}$$

$$(1-p')S_1'^2 + p'S_2'^2 = [(1-p)^5 + (1-p)^4 p]S_1^2 + 4(1-p)^4 p\left[S\left(\frac{3g_1^2 + 5g_1g_2}{5g_1+3g_2}\right)\right]^2 + \ldots \equiv G(p,S_1,S_2) \tag{12.b}$$

$$(1-p')S_1'^3 + p'S_2'^3 = [(1-p)^5 + (1-p)^4 p]S_1^3 + 4(1-p)^4 p\left[S\left(\frac{3g_1^2 + 5g_1g_2}{5g_1+3g_2}\right)\right]^3 + \ldots \equiv H(p,S_1,S_2) \tag{12.c}$$

where $S_i \equiv S(g_i)$ and $S'_i \equiv S(g'_i)$ $(i=1,2)$. The solution of the set of Eqs. (12) is given by

$$p' = \frac{L^2}{1+L^2} \tag{13.a}$$

$$S'_1 = F \pm L\sqrt{K} \tag{13.b}$$

$$S'_2 = F \mp \frac{1}{L}\sqrt{K} \tag{13.c}$$

where

$$K \equiv G - F^2 \geq 0 \tag{14}$$

and

$$L \equiv \frac{\sqrt{(H-3FK-F^3)^2 + 4K^3} - (H-3FK-F^3)}{2K^{3/2}} \tag{15}$$

The upper (lower) sign in Eqs. (13.b) and (13.c) is to be used in the region $S_1 > S_2$ $(S_1 < S_2)$, i.e. $g_1 > g_2$ $(g_1 < g_2)$. Eqs. (13) unambiguously provide $p'$, $S'_1$ and $S'_2$ as functions of $p$, $S_1$ and $S_2$ (or equivalently $p'$, $g'_1$ and $g'_2$ as function of $p$, $g_1$ and $g_2$; the reference conductance $g_0$ is cancelled out everywhere due to the homogeneous structure of Eqs. (13)), thus formally closing the operational problem. Finally, the conductivity $\sigma$ of the system, as a function of $p$ and $g_1/g_2$ for say fixed $g_2$, renormalizes as $1/g_2$ (see Refs. [3,15,16,20,21]).

## III RESULTS

The recursive relations (13) provide the surface indicated in Fig. 2. We note that:

i) two fully stable fixed points exist, namely $(p,S_1,S_2)=(0,0,0)$ and $(1,1,1)$, which enable (through the determination of the separatrix between their respective attractive bassins) the numerical calculation of the surface we are interested in;

ii) the insulator-resistor (superconductor-resistor) particular case corresponds to the lines on the $S_1=0$ and $S_2=0$ ($S_1=1$ and $S_2=1$) planes;

iii) the homogeneous or pure case $(g_1=g_2)$ corresponds to the twisted H-like line constituted by the $p=0$, $p=1$ and $s_1=s_2$ segments;

iv) the equal-concentration case $(p=1/2;\ g_1\neq g_2)$ corresponds to the line $S_1+S_2=1$.

In Fig. 3 we have represented, in the $(\sigma,p)$ space for fixed $g_2$ and typical values of $\alpha\equiv g_1/g_2\geq 0$, the surface appearing in Fig. 2.

The present $S$-RG provides the following *exact* results:

$$\left.\frac{d\sigma(p)}{\sigma(1)dp}\right|_{p=0}=\frac{2\alpha(1-\alpha)}{1+\alpha} \tag{16}$$

and consistently

$$\left.\frac{d\sigma(p)}{\sigma(1)dp}\right|_{p=1}=\frac{2(1-\alpha)}{1+\alpha} \tag{17}$$

as well as

$$\sigma(0)/\sigma(1) = \alpha \tag{18}$$

and

$$\frac{\sigma(p)}{\sigma(1)}\frac{\sigma(1-p)}{\sigma(1)} = \alpha \quad (\forall p) \tag{19}$$

hence

$$\sigma(1/2)/\sigma(1) = \sqrt{\alpha} \tag{19'}$$

Eq. (16) recovers the d=2 Eq. (9) of Ref. [4]; Eq. (19) recovers Eq. (5) of Ref. [9]. The q-RG is numerically less performant: for instance, instead of the *exact* Eq. (16), it yields

$$\left.\frac{d\sigma(p)}{\sigma(1)\,dp}\right|_{p=0} = \frac{8\alpha(1-\alpha)}{3\alpha+5} \tag{20}$$

which coincides with the (approximate) $(d,n)=(2,1)$ Eq. (8) of Ref. [4].

The critical exponents t and s (defined, in the $p \to p_c = 1/2$ bond percolation limit, through $\sigma(p;\alpha=0) \propto (p-p_c)^t$ and $\sigma(p;\alpha=\infty) \propto (p-p_c)^{-s}$) coincide[9] for the square lattice, but their exact numerical value is still unknown. To treat them within the present S-RG we calculate the Jacobian $J \equiv \partial(p',\alpha',S_2')/\partial(p,\alpha,S_2)$ at the percolation point $(p,\alpha,S_2)=(1/2,0,1)$, and obtain

$$J = \begin{pmatrix} \frac{13}{8} & 0 & 0 \\ 0 & 23/12 & 0 \\ \frac{13}{8} & 0 & 23/12 \end{pmatrix} \tag{21}$$

whose eigenvalues are $\lambda_1=13/8$ and $\lambda_2=\lambda_3=23/12$. The thermal critical exponent $\nu=\ell n b/\ell n\lambda_1$ as well as the exponent $t=s=\ell n\lambda_3/\ell n\lambda_1$ we obtain are indicated in Table I.

Before closing this section, let us heuristically propose an approximate *analytic* expression for $\sigma(p)/\sigma(1)$. Following along the lines of Refs. [27,28,29,30] we propose

$$\langle S\rangle_p = 1/2 \tag{22}$$

hence

$$(1-p)S_1 + pS_2 = 1/2 \tag{23}$$

therefore

$$\frac{(1-p)\alpha}{\alpha+\sigma(p)/\sigma(1)} + \frac{p}{1+\sigma(p)/\sigma(1)} = \frac{1}{2} \tag{24}$$

and consequently

$$\frac{\sigma(p)}{\sigma(1)} = \frac{1}{2}\left[\sqrt{(1-\alpha)^2(1-2p)^2+4\alpha} - (1-\alpha)(1-2p)\right] \tag{25}$$

This expression satisfies *all the available* **exact** *relations*, namely Eqs. (16)-(19). Nevertheless *it is not exact* as it leads to

$$\frac{\sigma(p)}{\sigma(1)} = 2p-1 \tag{26}$$

for $\alpha=0$, and

$$\frac{\sigma(p)}{\sigma(1)} = \frac{1}{2p-1} \tag{27}$$

for $\alpha \to \infty$, consequently t=s=1 which is wrong (see Table I). However, not to close to the critical region ($p \simeq 1/2$ and $\alpha \to 0$ or $\alpha \to \infty$) Eq.(25) should be a numerically quite reliable approximation for $\sigma(p)/\sigma(1)$.

## IV CONCLUSION

Within a real-space renormalization-group framework, we have calculated, for arbitrary concentrations and values of the (two) possible conductances, the conductivity of a square-lattice quenched bond-random resistor network with a binary distribution of conductances. The results are very encouraging as our best renormalization group (namely the S-RG) recovers *all* the available exact information (critical percolation probability, slopes, dual relations) and a satisfactory value for the insulator-resistor and superconductor-resistor mixtures critical exponents $t=s \simeq 1.340$ (to be compared with other recent numerically reliable values such as 1.26[23], 1.28[24], 1.30[25], 1.33..[26]). In some sense, such a high accuracy is not normally expected for a renormalization approach using such a small cluster (b=2). Three reasons converge for this fact to happen: (i) both clusters of Fig. 1 are self-dual (two-rooted) graphs, a choice which since long is known[31-33] to be very convenient for the square lattice; (ii) the renormalization space is relatively large in the sense that it is three-dimensional $(p, S_1, S_2)$; (iii) last but not least, the averages are performed on a very convenient variable (namely the S-variable) as it transforms, under duality, as simply as a probability (see Eq. (7)). An interesting technical point is worthy to be noted: the exact critical probability $p_c$=1/2 has been obtained *without* imposing a priori a *pure* percolation renormalization group recursive relation as usually done (see, for instance, Ref. [15]).

Finally, we have proposed, on heuristic grounds, a simple *analytic* expression (Eq. (25)) for the conductivity which, similarly to the present S-RG approach, also recovers *all* the available exact information. Excepting for the critical region (where it fails in reproducing satisfactory values for t=s), this expression is believed to be numerically quite reliable.

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CAPTION FOR FIGURES AND TABLES

Fig. 1 - Two-terminal self-dual arrays of conductances (o and ● respectively denote terminal and internal nodes). Within the present RG cluster (b) is renormalized into cluster (a).

Fig. 2 - RG flow in the $(p,S_1,S_2)$ space. The separatrix (surface delimited by the heavy lines) between the $g_1$-dominated and the $g_2$-dominated regions is invariant under the $(p,S_1,S_2)\leftrightarrow(1-p,S_2,S_1)$ transformation; the $p=1/2$ line constitutes an invariant sub-space corresponding to the equal concentration model. o, ● and ■ respectively denote fully stable, fully unstable and semi-stable fixed points.

Fig. 3 - Concentration dependence of the quenched bond-mixed resistor square lattice conductivity, for typical ratios $g_1/g_2$ (numbers on curves). $g_1/g_2=0$ and $g_1/g_2=\infty$ respectively correspond to the resistor-insulator and resistor--superconductor mixtures. The dashed line indicates the $p=1/2$ asymptot.

Table 1 - Present RG and other available values for the critical exponents $\nu$ and $t=s$.

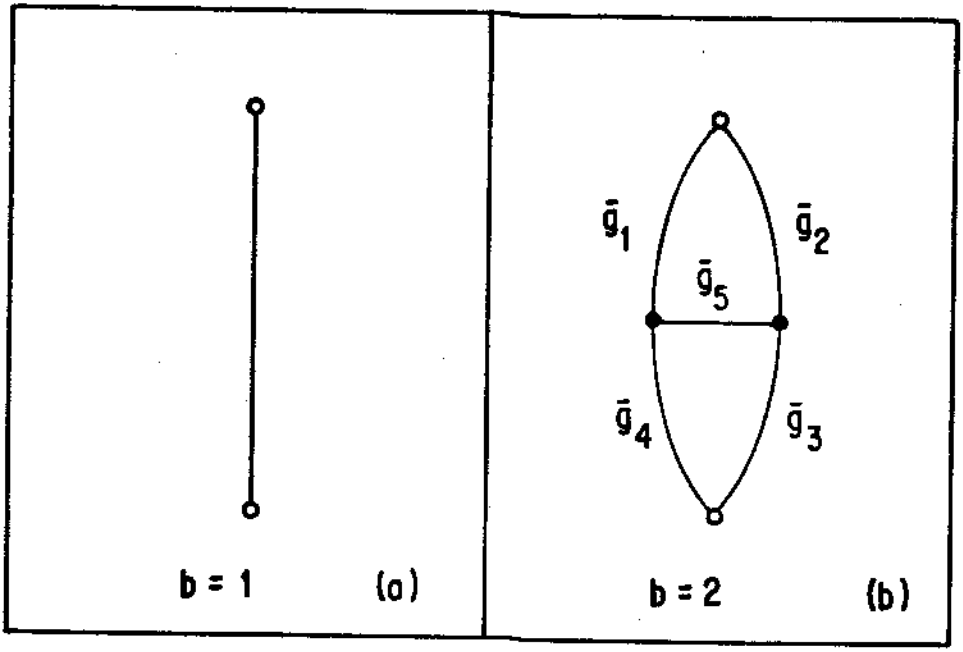


FIG.1

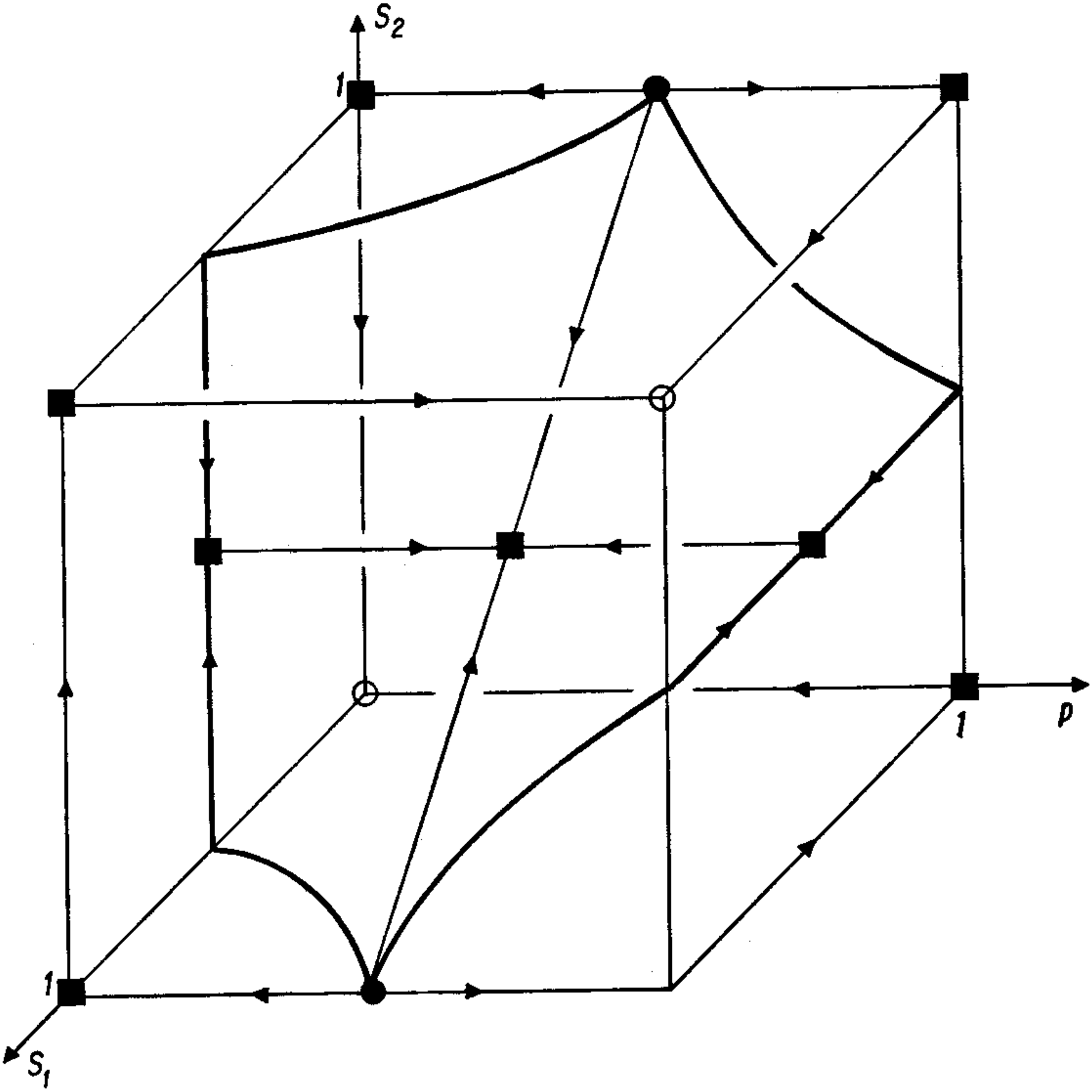


FIG.2

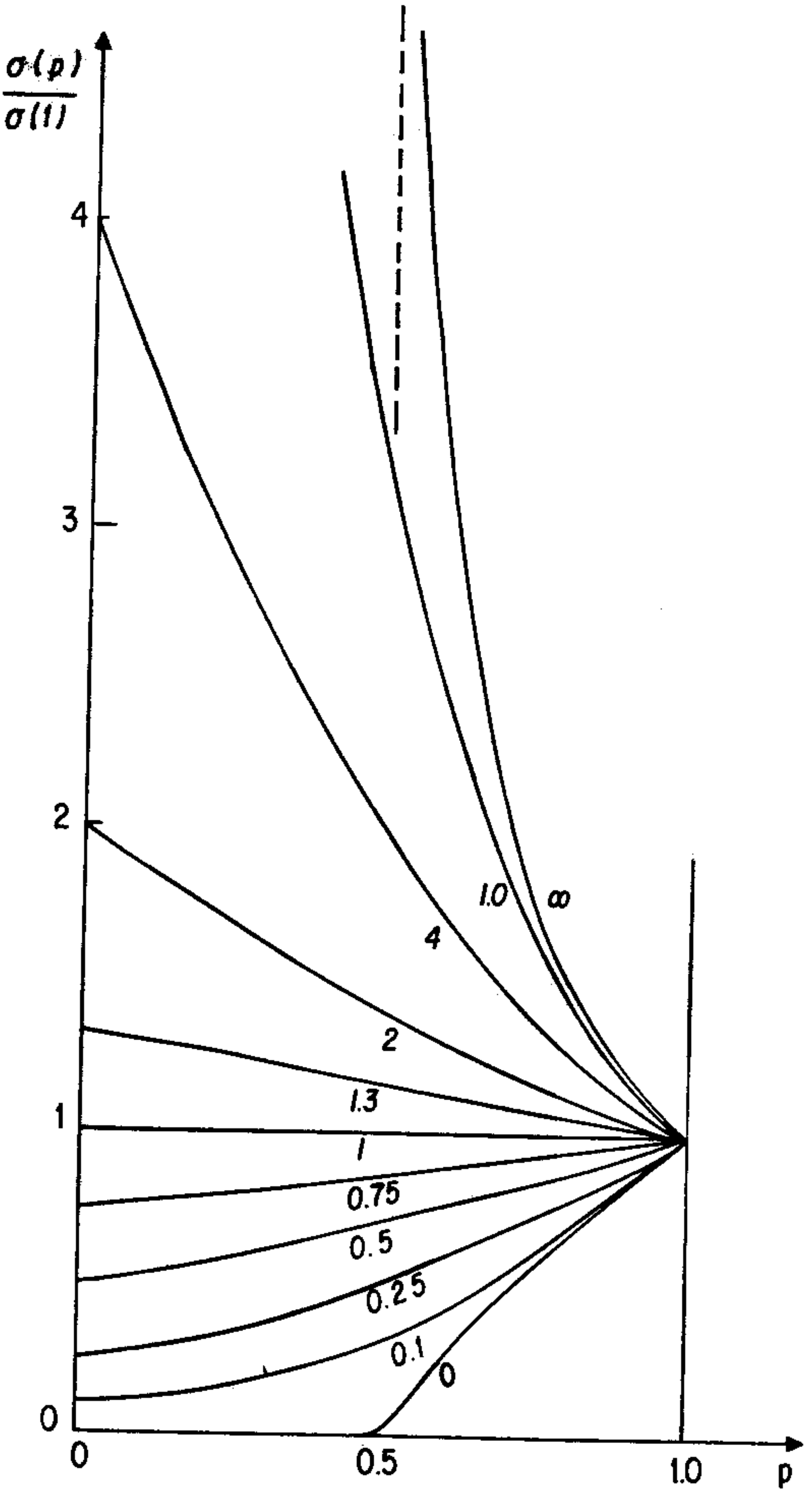


FIG.3

TABLE I

| | g-RG | s-RG | others |
|---|---|---|---|
| ν | 1.428 | 1.428 | 4/3 (exact[22]) |
| t | 1.235 | 1.340 | 1.26[23]<br>1.28 ± 0.03[24]<br>1.30[25]<br>4/3[26]<br>1.237[15] |

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