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ON THE DYNAMICS OF NON-HOLONOMIC SYSTEMS: THE CONSTRUCTION OF A LAGRANGIAN AND A HAMILTONIAN

by

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ABSTRACT

We show that once the motion of a non-holonomic system is known it is possible to reduce the system to the holonomic form. A (singular) Lagrangian function and a Hamiltonian which correctly describe the dynamics of the system can be constructed. The procedure we developed is applied to a well known system.

1- INTRODUCTION

The Lagrangian description of a mechanical system is based on the knowledge of the Lagrangian function which is supposed to contain all the physically relevant informations on the system. In general the systems which occur in nature are subjected to forces of constraints. Mathematically this means that in order to give a correct description of the evolution of the system one must take in account a certain number of relations among coordinates, velocities and time which express the existence of the forces of constraints. Those functions are known as constraint functions or, simply, constraints.

There are several kinds of constraints. Among those we will consider two very important classes. Denoting by $q_\alpha(t)$ and $\dot{q}_\alpha(t)$ the generalized coordinates and velocities, $\alpha=1\ldots,N$, we say that the constraints are holonomic or geometric constraints if they can be expressed as $K<N$ equations of the form

$$\phi_i(q,t) \equiv \phi_i(q_1,\ldots,q_N,t) = 0 , \quad i=1,\ldots K . \tag{1.1}$$

General velocity-dependent or kinematic constraints are expressed by equations of the type,

$$\psi_i(q,\dot{q},t) \equiv \psi_i(q_1,\ldots,q_N;\dot{q}_1,\ldots\dot{q}_N,t) = 0 , \quad i=1,\ldots,K . \tag{1.2}$$

When equations (1.2) cannot be reduced to the form (1.1) we say that the constraints are non-holonomic. [Neimark and Fufaev

(1972); Saletan and Cromer (1971)]

An important point on the theory of constrained systems is the question of the existence of an action principle. It is known [Saletan and Cromer (1970)] that the equations of motion for such systems can be obtained using variational techniques both for holonomic and non-holonomic systems, the difference in approach laying in the choice of the comparison paths. The results so far accepted can be summarized as follow. Let $L \equiv L(q_\alpha, \dot{q}_\alpha, t) \equiv L(q, \dot{q}, t)$ denote the Lagrangian function for the system when there are no constraints present which we call the free Lagrangian. The corresponding Euler-Lagrange vector will be denoted by

$$\Lambda_\alpha \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha}, \quad \alpha = 1, \ldots, N . \tag{1.3}$$

The dynamical evolution of the system under the influence of the constraints is given by (1.1) and

$$\Lambda_\alpha = \lambda^i \frac{\partial \phi_i}{\partial q^\alpha} \tag{1.4}$$

for holonomic system, and by (1.2) and

$$\Lambda_\alpha = \lambda^i \frac{\partial \psi_i}{\partial \dot{q}^\alpha} \tag{1.5}$$

for non-holonomic systems. We use the convention of summing over repeated indices and in the above expressions λ^i are

Lagrange multipliers. As it is known both cases can be dealt with in a unified way by using $\dot{\phi}_i=0$ instead of $\phi_i=0$ as the constraint equations in the holonomic case.

The point to be emphasized is that we do not have a Lagrangian function $\bar{L}(q,\dot{q},t)$ which completely describe the dynamics of the system including the informations concerning the existence of the constraints. Consequentely we do not have an associated action principle either.

The existence of such a Lagrangian function is obviously desirable not only from the classical point of view for it would enable one to quantize the system employing well known procedures. For holonomic systems it is possible to construct a Lagrangian function. Indeed, it is given by

$$\bar{L} = L + \lambda^i\phi_i \tag{1.6}$$

and the associated action principle leads to the correct equations of motion and the constraint equations.

It is usually accepted [Saletan and Cromer (1970); Gomes and Lobo (1979)] that for non holonomic systems it is not possible to construct such a Lagrangian function so that in this sense there does not exist an action principle for such systems.

The purpose of this paper is to make some developments about the existence of a Lagrangian function for non holonomic systems. We will show that once the motion of a non-holonomic system is known, it is possible to construct a Lagrangian function for the system. This Lagrangian function will correctly describe the dynamics of the system. With the Lagrangian so constructed we will show how to pass to the Hamiltonian formalism thus providing one with a consistent framework for canonical quantization. We shall not be concerned with the construction of an action principle for non-holonomic systems. This subject is presently under investigation. The paper is organized as follows: In section 2 we formally analyze the existence of an action principle for constrained systems. In section 3 we discuss the meaning of the integrability conditions for non-holonomic constraints and show how to construct the Lagrangian function for such systems. Sections 4 is devoted to an application to a well known system; some details are presented in order to clarify the method we developed. The Hamiltonian formalism is considered in sections 5 and 6. Final comments are in section 7.

2- THE ACTION PRINCIPLE FOR CONSTRAINED SYSTEMS

Given a non-holonomic system our concern is directed to the question: Can equations (1.2) and (1.5) be obtained from a variational principle $\delta\int \bar{L}dt=0$? We understand that the best way to look for an answer to this question is to analyze it from the point of view of the Helmohltz conditions [Engels (1975)]. Equations (1.2) and (1.5) are obtained under the hypothesis that system in study is described by a free Lagrangian function $L(q,\dot{q},t)$ and the constraint equations

$\psi_i(q,\dot{q},t)=0$. (*). The Euler-Lagrange vector (1.3) corresponding to the free Lagrangian L can be written as,

$$\Lambda_\alpha \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} = B_{\alpha\beta}(q,\dot{q},t)\ddot{q}^\beta + C_\alpha(q,\dot{q},t) \, . \qquad (2.1)$$

The functions $B_{\alpha\beta}(q,\dot{q},t)$ and $C_\alpha(q,\dot{q},t)$ are required to satisfy the following conditions [Engels (1975)]:

$$B_{\alpha\beta} \equiv B_{\beta\alpha} \, , \qquad (2.2a)$$

$$\frac{\partial B_{\alpha\beta}}{\partial \dot{q}^\nu} \equiv \frac{\partial B_{\nu\beta}}{\partial \dot{q}^\alpha} \, , \qquad (2.2b)$$

$$\frac{\partial C_\alpha}{\partial \dot{q}^\beta} + \frac{\partial C_\beta}{\partial \dot{q}^\alpha} \equiv 2\left(\frac{\partial B_{\alpha\beta}}{\partial q^\nu}\dot{q}^\nu + \frac{\partial B_{\alpha\beta}}{\partial t}\right) , \qquad (2.2c)$$

$$\frac{\partial B_{\alpha\beta}}{\partial q^\nu} - \frac{\partial B_{\nu\beta}}{\partial q^\alpha} \equiv \frac{1}{2}\left(\frac{\partial^2 C_\alpha}{\partial \dot{q}^\beta \partial \dot{q}^\nu} - \frac{\partial^2 C_\nu}{\partial \dot{q}^\beta \partial \dot{q}^\alpha}\right) \qquad (2.2d)$$

$$\frac{\partial C_\alpha}{\partial q^\beta} - \frac{\partial C_\beta}{\partial q^\alpha} \equiv \frac{1}{2}\left[\left(\frac{\partial^2 C_\alpha}{\partial q^\nu \partial \dot{q}^\beta} - \frac{\partial^2 C_\beta}{\partial q^\nu \partial \dot{q}^\alpha}\right)\dot{q}^\nu + \frac{\partial^2 C_\alpha}{\partial t \partial \dot{q}^\beta} - \frac{\partial^2 C_\beta}{\partial t \partial \dot{q}^\alpha}\right]. \qquad (2.2e)$$

Now consider the case when there are constraints. We denote by Q^A the set (q^α,λ^i) with the convention $Q^A \equiv q^\alpha$, for $A=\alpha=1,\dots,N$, and $Q^A \equiv \lambda^i$, for $A=i=N+1,\dots,N+K$. Denoting by $\bar{L}(Q,\dot{Q},t)$ the Lagrangian function associated with the system, the cor-

(*) In what follows we use the following conventions: greek indices $\alpha,\beta,\nu,\dots=1,\dots,N$; latin indices $i,j,k,\dots=N+1,\dots,N+K$ and capital latin indices $A,B,C,\dots=1,2,\dots,N+k$.

responding Euler-Lagrange vector is

$$\bar{\Lambda}_A \equiv \frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{Q}^A} - \frac{\partial \bar{L}}{\partial Q^A} = \bar{B}_{AB}(Q,\dot{Q},t)\ddot{Q}^B + \bar{C}_A(Q,\dot{Q},t) \ . \qquad (2.3)$$

The functions $\bar{B}_{AB}(Q,\dot{Q},t)$ and $\bar{C}_A(Q,\dot{Q},t)$ must satisfy the following conditions:

$$\bar{B}_{AB} \equiv \bar{B}_{BA} \ , \qquad (2.4a)$$

$$\frac{\partial \bar{B}_{AB}}{\partial Q^C} \equiv \frac{\partial \bar{B}_{CB}}{\partial Q^A} \ , \qquad (2.4b)$$

$$\frac{\partial \bar{C}_A}{\partial \dot{Q}^B} + \frac{\partial \bar{C}_B}{\partial \dot{Q}^A} \equiv 2\left(\frac{\partial \bar{B}_{AB}}{\partial Q^C}\dot{Q}^C + \frac{\partial \bar{B}_{AB}}{\partial t}\right) \ , \qquad (2.4c)$$

$$\frac{\partial \bar{B}_{AB}}{\partial Q^C} - \frac{\partial \bar{B}_{CB}}{\partial Q^A} \equiv \frac{1}{2}\left(\frac{\partial^2 \bar{C}_A}{\partial \dot{Q}^B \partial \dot{Q}^C} - \frac{\partial^2 \bar{C}_C}{\partial \dot{Q}^B \partial \dot{Q}^A}\right) \ , \qquad (2.4d)$$

$$\frac{\partial \bar{C}_A}{\partial Q^B} - \frac{\partial \bar{C}_B}{\partial Q^A} \equiv \frac{1}{2}\left[\left(\frac{\partial^2 \bar{C}_A}{\partial Q^C \partial \dot{Q}^B} - \frac{\partial^2 \bar{C}_B}{\partial Q^C \partial \dot{Q}^A}\right)\dot{Q}^C + \frac{\partial^2 \bar{C}_A}{\partial t \partial \dot{Q}^B} - \frac{\partial^2 \bar{C}_B}{\partial t \partial \dot{Q}^A}\right] . \qquad (2.4e)$$

According to equations (1.2) and (1.5) we can assure that $\bar{L}$ is the Lagrangian function for the constrained system if we impose that

$$\bar{\Lambda}_\alpha \equiv B_{\alpha\beta}\ddot{q}^\beta + C_\alpha - \lambda^i \frac{\partial \psi_i}{\partial \dot{q}^\alpha} = 0 \ , \qquad (2.5)$$

$$\bar{\Lambda}_j = 0 \ , \qquad (2.6)$$

The above conditions express the fact that $\bar{L}(Q,\dot{Q},t)$ lead to the equations of motion, equations (2.5), and the constraint equations, equations (2.6)

Equations (2.5,6) can be viewed as conditions to be imposed on the functions $\bar{B}_{AB}$ and $\bar{C}_A$. Using (2.3) these conditions are:

$$\bar{B}_{AB} \equiv B_{\alpha\beta} , \qquad (A=\alpha, B=\beta; \alpha,\beta=1,\dots,N) \qquad (2.7)$$

$$\bar{B}_{AB} \equiv 0 , \qquad (A \text{ or } B=N+1,\dots,N+K) \qquad (2.8)$$

$$\bar{C}_A \equiv C_\alpha - \lambda^i \frac{\partial\psi_i}{\partial\dot{q}^\alpha} , \qquad (A=\alpha=1,\dots,N) \qquad (2.9)$$

$$\bar{C}_A \equiv C_j = 0 , \qquad (A=j=N+1,\dots,N+K) \qquad (2.10)$$

The problem now is reduced to the validity of the system (2.4a-e) restricted by the conditions (2.2a-e) and (2.7-10).

Now, conditions (2.4a,b) are trivially verified while conditions (2.4c) require that

$$\frac{\partial^2\psi_i}{\partial\dot{q}^\alpha\partial\dot{q}^\beta} = 0 , \qquad (2.11)$$

$$\frac{\partial\bar{C}_j}{\partial\dot{q}^\beta} = 0 . \qquad (2.12)$$

The conditions (2.4d) are verified with no aditional restrictions (*) while (2.4e) requires that

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(*) We used (2.12) and the fact the functions $\bar{C}_j(q,\dot{q},t)$ do not depend explicitly on λ^i.

$$\frac{\partial^2 \psi_i}{\partial q^\alpha \partial \dot{q}^\beta} = \frac{\partial^2 \psi_i}{\partial q^\beta \partial \dot{q}^\alpha} , \tag{2.13}$$

$$\frac{\partial \bar{C}_j}{\partial q^\beta} + \frac{\partial \psi_j}{\partial \dot{q}^\beta} = 0 . \tag{2.14}$$

Let us consider the meaning of these results. From equations (2.12) it follows that the functions $\bar{C}_j$ are not dependent on the generalized velocities, i.e., $\bar{C}_j = \bar{C}_j(q,t)$. Equations (2.11) require the constraint functions to be at most linear functions on the generalized velocities. Equations (2.13) are the integrability conditions for the constraints $\psi_i(q,\dot{q},t)=0$, and they assure the existence of a set of functions $g_i(q,t)=$ constant, such that $\psi_i \equiv \dot{g}_i$. Finally, it follows from (2.14) that $g_i = \bar{C}_i$.

Thus, there will exist a Lagrangian function (and an associated variational principle) for constrained systems if the constraint equations are linear functions of the generalized velocities and can be reduced to the holonomic form. One can go a little far and write an explicit form for the Lagrangian function corresponding to these cases. For instance, using the procedure described by Engels [Engels (1975)] one obtains:

$$\bar{L} = \bar{L} + \lambda^i \bar{C}_i . \tag{2.15}$$

3- THE REDUCTION OF NON-HOLONOMIC CONSTRAINTS TO THE HOLONOMIC FORM.

From the results of the last section one concludes that it is possible to construct a Lagrangian function and a corresponding action principle which lead to the equations of motion and constraint equations only for holonomic systems.

Let us analyze this statement in some detail. For simplicity we consider a system subjected to only one constraint equation and write it as (*)

$$\Omega \equiv X_\alpha(q)\ dq^\alpha = 0\,, \tag{3.1}$$

and take N=3. The integrability condition for (3.1) is

$$\vec{X}.\ \text{rot}\ \vec{X} = 0\ , \qquad \vec{X} \equiv (X_1, X_2, X_3) \tag{3.2}$$

If this condition is fulfiled then there exists a function (an integrating factor), say M(q), such that MΩ is an exact differential (**).

Geometrically the fulfilment of the integrability conditions mean , for a given initial configuration, the existence

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(*) The form of equation (3.1) does not introduce any essential restriction in the present investigation.

(**) This conclusion also hold for N≠3. We specialize for N=3 only for simplicity. The basic results we will obtain are also valid for the general case N≠3. See [Forsyth, 1903].

of points in configuration space which are not accessible to the system by trajectories satisfying (3.1). The converse of this statement is also true and is just Caratheodory theorem [Buchdahl 1949)].

Now, suppose that condition (3.2) does not hold and thus Ω represents a non-holonomic constraint. What can be concluded is the non existence of a single function, say $\phi(q)$, such that $d\phi = N(q)\Omega$, where $N(q)$ is an integrating factor. Of course this by no means imply that the equation $\Omega=0$ does not admit solutions. Actually, it is well known that if we choose an arbitrary function,

$$\chi(q) = 0 \tag{3.3}$$

it is possible to determine another function

$$\psi(q) = \text{constant} = c \tag{3.4}$$

such that (3.3) and (3.4) represent a solution for equation (3.1). In fact, from (3.3) we can write

$$d\chi = 0 \tag{3.3.1}$$

so that when the form $\chi(q)$ is specified we can use (3.3) and (3.3.1) to determine q_3 and dq_3 (for instance) in terms of the others q_i and dq_i. After substituting these relations in (3.1) the result will be a two dimensional differential equation wich can always be integrated to obtain a solution of the form (3.4). (For details see [Forsyth, 1903].)

Now, assigning to $\chi(q)$ every possible forms we obtain the whole set of possible solutions. These solutions represent a family of curves each one of them being a solution of equations (3.1).

This result admits a physical interpretation, namely, that one non-holonomic constraint equation can be substituted by two holonomic constraint equations according to the procedure described above. The question resides on the choice of the function $\chi(q)$. From the mathematical point of view the function $\chi(q)$ is arbitrary but this is clearly not so from the physical viewpoint. Among the whole set of mathematically admissible functions there is only one which minimize the action functional, the surface where the motion of the system actually occur. Therefore, this must be the surface $\chi(q)=0$ in agreement with the basic postulate of Classical Mechanics, the Least Action Principle. Once this function is known, one can guess which is the corresponding $\psi(q)$ function for the problem at hand. These functions will behave like two holonomic constraints which will substitute for the original non-holonomic constraint therefore, it will be possible to construct a new Lagrangian function $\bar{L}$ for the system. This Lagrangian function $\bar{L}$ will contain all the phisically relevant informations, about the system including the constraints.

One can argue about the reasons for constructing a Lagrangian function after the motion of the system is known. The point is that the form of the Lagrangian function $\bar{L}$ we obtain is exactly the same as expressions (2.15) and it enables us to construct a Hamiltonian for the system. Therefore, the standard quantization procedures can be used. From our point of view this result justifies the efforts to construct $\bar{L}$.

In the following section we apply this method to a well known non-holonomic system. The aim is to show how it works in pratice and call attention to some points where it can be simplifyed.

4- APPLICATION: A ROLLING DISK CONSTRAINED TO REMAIN VERTICAL

We consider the motion of a sharp edge homogeneous disk of mass m and radius R that rolls without slipping on a perfectly rough horizontal plane and is constrained to remain vertical. This is a well known problem. [Saletan and Cromer (1970), Neimark and Fufaev (1972), Whittaker (1936)]. The generalized coordinates are choosen as follow: q_1 and q_2 are the projection of the center of mass on the horizontal plane, q_3 is the angle between the plane of the disk and the q_1-axe; q_4 is the angle between a diameter of the disk and a vertical line.

The free Lagrangian function for the system is:

$$L = \frac{1}{2}m(\dot{q}_1^2+\dot{q}_2^2) + \frac{1}{2}I_1\dot{q}_3^2 + \frac{1}{2}I_0\dot{q}_4^2 \ , \tag{4.1}$$

where I_0 is the moment of inertia of the disk with respect to an axe passing through its center and I_1 is the moment of inertia with relation to a diameter.

The constraints for the system can be expressed by the following equations:

$$\phi_1 = R\dot{q}_4\cos q_3 - \dot{q}_1 = 0 \ , \qquad (4.2)$$

$$\phi_2 = R\dot{q}_4\sin q_3 - \dot{q}_2 = 0 \ . \qquad (4.3)$$

The corresponding integrability conditions are not satisfied, hence equations (4.2,3) represents two non-holonomic constraints. The equations of motion obtained by the standard procedure described in section 1 are

$$m\ddot{q}_1 = -\lambda_1 \ , \qquad (4.4a)$$

$$m\ddot{q}_2 = -\lambda_2 \ , \qquad (4.4b)$$

$$I_1\ddot{q}_3 = 0 \ , \qquad (4.4c)$$

$$I_0\ddot{q}_4 = \lambda_1 R\cos q_3 + \lambda_2 R\sin q_3 \ , \qquad (4.4d)$$

which must be supplemented by the constraints (4.2,3). The Lagrangian multipliers λ_1 and λ_2 can be eliminated from these equations. One obtains $\lambda_1 = mR\dot{q}_3\dot{q}_4\sin q_3$, $\lambda_2 = -mR\dot{q}_3\dot{q}_4\cos q_3$. Using these values we can rewrite equations (4.4a,d) as

$$\ddot{q}_1 = -R\dot{q}_3\dot{q}_4\sin q_3 \qquad (4.5a)$$

$$\ddot{q}_2 = R\dot{q}_3\dot{q}_4\cos q_3 \qquad (4.5b)$$

$$\ddot{q}_3 = 0 \qquad (4.5c)$$

$$\ddot{q}_4 = 0 \qquad (4.5d)$$

Now, the solutions for equations (4.5a,d) and (4.2,3) corresponding to arbitrary initial data, $\vec{q}_0=(q_{10},q_{20},q_{30},q_{40})$, $\dot{\vec{q}}_0=(\dot{q}_{10},\dot{q}_{20},\dot{q}_{30},\dot{q}_{40})$, are

$$q_1 = a+R\frac{\dot{q}_{40}}{\dot{q}_{30}}\sin(\dot{q}_{30}t+q_{30}) , \tag{4.6a}$$

$$q_2 = b-R\frac{\dot{q}_{40}}{\dot{q}_{30}}\cos(\dot{q}_{30}t+q_{30}) , \tag{4.6b}$$

$$q_3 = \dot{q}_{30}t+q_{30} , \tag{4.6c}$$

$$q_4 = \dot{q}_{40}t+q_{40} , \tag{4.6d}$$

where a and b are two constants.

It is worthwhile to observe that the constraint equations can in general be expressed in several different ways. For the problem at hand it can be show [Saletan and Cromer, (1970) and (1971)] that they can be represented by the (non linear) equations

$$\phi'_1 = \dot{q}_1^2+\dot{q}_2^2-R^2\dot{q}_4^2 = 0 , \tag{4.7a}$$

$$\phi'_2 = \dot{q}_1\sin q_3-\dot{q}_2\cos q_3 = 0 . \tag{4.7b}$$

It is also possible to express the constraints by a single equation [Whittaker, (1936)]

$$\phi = \dot{q}_1\tan q_3-\dot{q}_2 = 0 . \tag{4.8}$$

We shall use this later form to express the constraints. The corresponding equations of motion are

$$m\ddot{q}_1 = \lambda \tan q_3 \ , \qquad (4.9a)$$

$$m\ddot{q}_2 = -\lambda \ , \qquad (4.9b)$$

$$\ddot{q}_3 = 0 \ , \qquad (4.9c)$$

$$\ddot{q}_4 = 0 \ , \qquad (4.9d)$$

which must be solved taking into account equation (4.8). For the Lagrange multiplier we obtain

$$\lambda = -m\dot{q}_1\dot{q}_3 \ .$$

Using this value for λ we can solve equations (4.9). We obtain

$$q_1 = a + \frac{u}{\dot{q}_{30}} \sin(\dot{q}_{30}t + q_{30}) \ , \qquad (4.10a)$$

$$q_2 = b + vt - \frac{u}{\dot{q}_{30}} \cos(\dot{q}_{30}t + q_{30}) \ , \qquad (4.10b)$$

$$q_3 = \dot{q}_{30} t + q_{30} \ , \qquad (4.10c)$$

$$q_4 = \dot{q}_{40} t + q_{40} \ , \qquad (4.10d)$$

where a,b,u and v are constants and no use has been made of condition[(*)] (4.8). Now, taking in account that condition we

(*) In order to obtain the explicit value for λ we used $\dot{\phi}=0$ instead of $\phi=0$.

get v=0. Thus, it follows from (4.10) that

$$(q_1-a)^2 + (q_2-b)^2 = \frac{u^2}{\dot{q}_{30}^2},$$

$$\dot{q}_1^2 + \dot{q}_2^2 = u^2 \quad .$$

One can easily verify that $u=R\dot{q}_{40}$ and so the disk moves with this constant speed in a circle of radius $R\dot{q}_{40}/\dot{q}_{30}$ centered at (a,b) [Saletan and Cromer, (1971)]. It also follows from the values of u and v that expressions (4.10a,d) reduces to the solutions (4.6a,d).

We now apply our method to this problem. For simplicity we set a=b=0. From the solutions (4.10a,d) it follows that the motion of the system takes place on the surface defined by the equation

$$\theta(q) \equiv q_1 + q_2 \tan q_3 = 0 \quad . \tag{4.11}$$

This surface must be taken as our $\chi(q)$ function, equation (3.3). Now, following the procedure described in the last section we obtain

$$\psi(q) = q_2 + c\cos q_3 = 0 \tag{4.12}$$

where c is a constant which depends on the initial data. Equations (4.11,12) are the holonomic constraints that substitute for the non-holonomic one given by equation (4.8). On the other hand equations (4.11,12) are equivalent to :

$$\bar{\chi}(q) \equiv q_1 - c\sin q_3 = 0 \qquad (4.11.a)$$

$$\psi(q) \equiv q_2 + c\cos q_3 = 0 \tag{4.13}$$

and we will use this last set of equations as the holonomic constraints corresponding to the non-holonomic system we are considering (*). Using these constraints we can write the Lagrangian functions $\bar{L}$ for the system:

$$\bar{L} = \frac{m}{2}(\dot{q}_1^2+\dot{q}_2^2) + \frac{1}{2}I_1\dot{q}_3^2 + \frac{1}{2}I_0\dot{q}_4^2 + \lambda_1(q_2+c\cos q_3) + \lambda_2(q_1-c\sin q_3) . \tag{4.14}$$

The Lagrangian function (4.14) carries all the relevant informations for the dynamical description of the system. Indeed, considering the Lagrange multipliers as additional coordinates the Euler-Lagrange equations that follows from (4.14) are

$$m\ddot{q}_1 = \lambda_2 , \tag{4.15a}$$

$$m\ddot{q}_2 = \lambda_1 , \tag{4.15b}$$

$$I_1\ddot{q}_3 = -c(\lambda_1\sin q_3 + \lambda_2\cos q_3) , \tag{4.15c}$$

$$\ddot{q}_4 = 0 , \tag{4.15d}$$

$$q_2 + c\cos q_3 = 0 , \tag{4.15e}$$

$$q_1 - c\sin q_3 = 0 . \tag{4.15f}$$

With the Lagrange multipliers given by

$$\lambda_1 = mc\dot{q}_3^2\cos q_3 , \qquad \lambda_2 = -mc\dot{q}_3^2\sin q_3 ,$$

(*)The set of equations (4.11,12) and (4.11.a,13) are, of course, equivalents. We choose to work with the second set because this will avoid many unecessary calculations in what follows.

the system of equations (4.15a,d) reduces to

$$\ddot{q}_1 = -c\dot{q}_3^2 \sin q_3 \ ,$$
$$\ddot{q}_2 = c\dot{q}_3^2 \cos q_3 \ ,$$
$$\ddot{q}_3 = 0 \ , \tag{4.16}$$
$$\ddot{q}_4 = 0 \ .$$

Solving this system one will arrive to the same solutions as given by expressions (4.6a-d) with a=b=0 (*).

5- THE HAMILTONIAN APPROACH TO NON-HOLONOMIC SYSTEM.

Once we have obtained the Lagrangian function associated with a non-holonomic system we can develop a Hamiltonian formalism. The procedure is essentially Dirac's theory of constrained systems [Dirac, (1964)] since now we have a (singular) Lagrangian function to describe the system. There is, however, some peculiarities which we will clarify in what follows. For definiteness let us consider a Lagrangian function of the form (1.6), which we rewrite,

$$\bar{L}(q,\dot{q},t) = L(q,\dot{q}.t) + \lambda^i \phi_i(q) \ , \tag{5.1}$$

(*) As we had pointe out the value of the constant c depends on the specification of the initial data. For the data corresponding to solutions (4.6a-d) it can be verified that c corresponds to the radius $\frac{R\dot{q}_{40}}{\dot{q}_{30}}$ of the circle described by the disk.

where the functions λ^i are Lagrange multipliers and $\phi_i(q)$ are the holonomic constraint functions which substitute for the non-holonomic constraint of the original problem. Now, the key step at this point is to treat the Lagrangian multipliers as additional generalized coordinates, thus formally enlarging the configuration space. The functions $\phi_i(q)$ can be considered as arbitrary functions in the sense that we do not need to consider them as constraints. This information will follow as a consequence of the theory.

In order to pass to the Hamiltonian formalism we define the momenta canonnicaly conjugated to the generalized coordinates (which now is the set $(q^\alpha, q^i \equiv \lambda^i)$):

$$p_\alpha = \frac{\partial \bar{L}}{\partial \dot{q}^\alpha} = \frac{\partial L}{\partial \dot{q}^\alpha} , \tag{5.2}$$

$$\pi_i = \frac{\partial \bar{L}}{\partial \dot{\lambda}^i} = 0 . \tag{5.3}$$

This last expression follows from the fact that there is no dependence of $\bar{L}$ on the "velocities" $\dot{\lambda}_i$.

Equations (5.3) are the primary constraints of the theory and must be written as weak equations,

$$\pi_i \approx 0 \tag{5.4}$$

Hence, the additional degrees of freedom we introduced are constrained by these equations. In general this means that the "coordinates" λ^i are arbitrary or otherwise determined and, as we shall see, this will be the case.

The canonical Hamiltonian for the system is

$$\bar{H}_c = p_\alpha \dot{q}^\alpha - \bar{L}$$

$$= p_\alpha \dot{q}^\alpha - L - \lambda^i \phi_i = H_c - \lambda^i \phi_i \quad , \qquad (5.5)$$

where H_c is the canonical Hamiltonian for the system when there are no constraints, i.e., the "free Hamiltonian". According to Dirac theory we must add to the Hamiltonian (5.5) a linear combination of the primary constraints (5.4) and impose the consistency conditions that those constraints are preserved in time. But as it is usual in theories where some momenta are constrained to be zero(*) we can freeze the momenta π_i considering equations (5.4) as strong equations.

Now, the consistency conditions for equations (5.4) lead immediately to

$$\phi_i(q) \approx 0 \quad , \qquad (5.6)$$

and we recover the information that the functions ϕ_i are the constraints of the theory.

We must continue the procedure and impose the time preservation of the (secondary) constraints (5.6). However, now we

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(*) This is the case, for instance, in the canonical formalism of the general theory of relativity where the momenta conjugated to the lapse and shift functions is constrained to be zero [Misner et al, (1973)]. See also [Dirac, (1950)] face a new situation. It happens (at least for the cases we have studied) that the second step beyond (5.6) leads to the determination of the functions λ^i as functions of the q^α's and p_α's. At this point the procedure must be stoped [Dirac (1950)]. There will remain a definite number of secondary constraints which are in fact second class. Now, what has to be done is to use the Dirac brackets with respect to these constraints and set them all strongly equal to zero. Therefore, the Hamiltonian we are left with is the free canonical Hamiltonian but the equations of motion are given in terms of Dirac brackets,

$$\dot{F} = \{F,H_c\}^* = \{F,H_c\} - \{F,\phi_i\}C^{-1}_{ij}\{\phi_j,H_c\} \qquad (5.7)$$

where C^{-1}_{ij} denotes the elements of the matrix inverse of $C=\|\{\phi_i,\phi_j\}\|$.

6- THE HAMILTONIAN APPROACH FOR THE ROLLING DISK

We now apply the method described in the last section to the problem we dealt with in section 4. The Lagrangian function is given by expression (4.14),

$$\bar{L} = \frac{1}{2}[m(\dot{q}_1^2+\dot{q}_2^2) + I_1\dot{q}_3^2 + I_0\dot{q}_4^2] + q_5\theta_1 + q_6\theta_2 \quad , \qquad (6.1)$$

where we used the notation

$$\lambda_1 = q_5 \ , \tag{6.1a}$$

$$\lambda_2 = q_6 \ , \tag{6.1b}$$

$$q_2 + c\cos q_3 = \theta_1 \ , \tag{6.1c}$$

$$q_1 - c\sin q_3 = \theta_2 \ . \tag{6.1d}$$

The corresponding canonical Hamiltonian is

$$\bar{H}_c = H_c - q_5\theta_1 - q_6\theta_2 \tag{6.2}$$

where H_c is the free canonical Hamiltonian

$$H_c = \frac{1}{2m}(p_1^2+p_2^2) + \frac{1}{2I_1}p_3^2 + \frac{1}{2I_0}p^2{}_4 \ . \tag{6.3}$$

The primary constraints are

$$\pi_1 \approx 0 \ , \qquad \pi_2 \approx 0 \ , \tag{6.4}$$

with π_i defined by equations (5.3).

The consistency conditions $\dot{\pi}_i \approx 0$ lead to

$$\theta_1 \approx 0 \ , \tag{6.5a}$$

$$\theta_2 \approx 0 \ . \tag{6.5b}$$

Imposing the time preservation of the secondary constraints

In order to write the Hamiltonian equations of motion we need the matrix C^{-1}, inverse of $C=\|\{\theta_i,\theta_j\}\|$. From (6.8) we obtain

$$\Delta \equiv \det\|\{\theta_i,\theta_j\}\| = \left(\frac{I_1+mc^2}{m^2 I_1}\right)^2 \neq 0\ , \tag{6.9}$$

A straightforward calculation leads to

$$C^{-1} = \frac{m^2 I_1}{I_1+mc^2}\left\|\begin{matrix} 0 & \frac{c^2 p_3}{I_1^2} & -(\frac{1}{m}+\frac{c^2}{I_1}\cos^2 q_3) & (\frac{c^2}{I_1}\sin q_3\cos q_3) \\ -\frac{c^2 p_3}{I_1^2} & 0 & \frac{c^2}{I_1}\sin q_3 \cos q_3 & -(\frac{1}{m}+\frac{c^2}{I_1}\sin^2 q_3) \\ (\frac{1}{m}+\frac{c^2}{I_1}\cos^2 q_3) & -(\frac{c^2}{I_1}\sin q_3\cos q_3) & 0 & 0 \\ -\frac{c^2}{I_1}\sin q_3\cos q_3) & (\frac{1}{m}+\frac{c^2}{I_1}\sin^2 q_3) & 0 & 0 \end{matrix}\right\| \tag{6.10}$$

We now use the Dirac brackets with respect to the secondary constraints $\{\theta_i\}$ and set all the constraints strongly equal to zero. The equation of motion for an arbitrary dynamical variable is given by

$$\dot{F} = \{F,H_c\}^* = \{F,H_c\} - \frac{mcp_3^2}{I_1^2}\left[\{F,\theta_1\}\cos q_3 - \{F,\theta_2\}\sin q_3\right]\ . \tag{6.11}$$

It is an easy task now to show that (6.11) leads to the same equations of motion as obtained before, namely, equations (4.16).

7- FINAL COMMENTS

In this paper we have stablished a procedure to transform a non-holonomic system into a equivalent holonomic system. A singular Lagrangian function associated with the equivalent holonomic system is written down based on the knowledge of the surface (a submanifold of the configuration space) where the motion actually occur. As we mentioned before we believe that the efforts to construct such a holonomic system are justified because, at least in principle, one can quantize such systems using well known procedures [Dirac, 1964; Fradkin and Vilkovisky, 1977].

We did not touch on the question of constructing an action functional for non-holonomic system which, as yet, is an open problem (*). Our procedure does not lead to any specific simplification of this problem. However, we expect that a deeper analysis might shed some light in the direction to be taken in order to overcome this question.

Finally, we mention that our procedure does not share any relation with the procedures proposed long ago by J.W. Campbell and others. (See [Campbell, 1936] and references there in).

(*) For details on the extension of the Least Action Principle to non-holonomic systems see the excellent paper by L.A. Pars, [Pars,1954].

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