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SEPARABLE ORTHOGONAL COORDINATES AND PARTICLE CREATION FOR AN ACCELARATING OBSERVER

by

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Abstract

We present an exactly solvable example of a non-stationary system, which has an inertial and an uniform accelerated asymptotic region. We construct a set of solutions that are quasi-classical in these two regions and compare the two sets. The Bogoliubov-coefficients have thermal character and show a temperature of $a_{\infty} / 2\pi$, where a_{∞} is the asymptotic acceleration in the out-region. This result is much what one would expect on the grounds of the Hawking-effect. It implies that the natural particle number is not conserved in free Minkowski-space.

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Key-words: Quantum field theory in curved space; Particle creation; Klein-Gordon equation; Separable coordinates; Non-stationary systems.

I - Introduction

This work has its place in the interface of quantum field theory and gravitation. It is well known that this is a dangerous jungle where common sense will help you little and where intuition is almost your only equipment. But neverthless this enterprise has to be tried, the hope to unify all the forces of nature in the context of a geometrical theory is very tempting. The dangers are there, because we have very few possibilities of experiments, because we seem to have to quantize the geometry itself, because we sure have to survive a lot of phylosophical revolutions, and we will not end this travel before the end of the crisis of the metaphysics[1].

In this scenario we choose a safe way and attack the very first problem of writing quantum field theory in general coordinates without leaving flat space-time. This is known, after the work of Fulling[2], to be anything but trivial: the very concept of particle is not well definable[3], so that what appears to be the vacuum to an inertial observer will look like a thermal state for an uniformly accelerated observer with a temperature proportional to his acceleration. In the technically very similar Hawking-effect the acceleration will be replaced by the surface gravity of a black-hole showing the first well stablished physical effect of quantum gravity[4].

We now mention some other papers that followed in order to situate our work in the literature. The work of Fulling is entirely based on the Bogoliubov-coefficient, that gives a measure of the amount of particles of the new field that are present in the old vacuum. This analysis was confirmed by the introduction of the more physical - but more elaborated - concept of particle detector by Unruh[5]. But soon the two methods contradicted one another with the study of uniformly rotating observers[6] where the expectation value of the number of particles in the inertial vacuum vanishes whereas the signal of a rotating detector does not. After that, Letaw and Pfautsch[7] showed that in the coordinates related to the 6 Killing vectorfields of the Minkowski space there are only two kinds of vacuum, but that the six immersed detectors respond all differently. Accidentally, it is exactly the systems that show a non-vanishing vacuum expectation value that have an event-horizon. People speculated about their relation, but the work of Sanchez[8] showed that they were wrong. In her work entropy is directly related to the presence of a coordinate singularity, and temperature is higher the stronger the singularity of the coordinate transformation. In a beautiful paper Grove and Ottewill[9] succeeded to explain the contradiction of expectation value and detector signal by separating the detectors' self-excitation from the net absorption effect. They also pointed out the problem of taking point detectors and suggested a criterium to construct privileged detectors that will not self-emmit. This recipe was misinterpreted by Mhyrvold[10] and was shown to be unmanageable in very simple situations in spite of its theoretically very beautiful features (in particular, it is not equivalent to Letaw and Pfautsch's coordinates in the case that the detector follows one Killing trajectory as they speculated[11]). People also wonder about energy conservation of the Fulling effect[12], about its anisotropy[13], about topological effects[14], and about the dependence of the results on the coupling to the detector field[15]. All this wirr-warr leads Hinton[16] to discuss the episthemological content of the detector concept, demonstrating its almost uselessness.

In the present paper we further investigate the two dimensional Minkowski space with the help of very geometrical coordinates: they are the separable orthogonal coordinates for the Klein-Gordon equation separates. They play the same important role as their euclidean equivalents - polar, elliptic and parabolic coordinates - play in all physics. These coordinates are adapted to very interesting physical situations like a global boost, an observer that is inertial in the past and uniformly accelerated in the future. They are also useful to study compact regions of Minkowski plane. They are often not static and, because of their simplicity, give us hope to master this difficult situation. They are also easily generalized to more dimensions. And above all they are such that the Klein-Gordon equation (and almost all other interesting quantities) are exactly solvable, destillating the physical understanding from the mathematical dificulties of the problem.

We deduce and classify the coordinates in the next section. We then succeed in solving the special case of an accelerating observer, thus giving an alternative to the detector analysis in interpreting the Fulling effect. This is done in the last two sections, where we calculate and discuss these results.

II - Separable orthogonal coordinates

We look for the orthogonal coordinates where the Klein-Gordon equation separates and is exactly solvable. Separable coordinates are well known in two- and three-dimensional Euclidian space since the work of Eisenhardt[17]. In Minkowski space they have also been classified by Urbantke[18] with elegant methods of projective geometry. We note that no classification exists in four dimensions; not even their number is known. Recently some work has been done on the more general question of separable manifolds[19]. Here we follow the reasoning of Morse, Feshbach[20] and get the coordinate transformation, metrics and all that, in a more usual language in two dimensional Minkowski space.

We begin with cartesian coordinates (t, x) and the metric

$$g = \eta_{ab} \, dx^{a} \, dx^{b} \qquad \text{where} \quad a, b, \ldots = 0 \text{ or } 1; \tag{1}$$

η has signature (+, -). In normalized light coordinates the metric has only one independent component,

$$\begin{cases} l^{-} = 2^{-1/2} \, (t + x) \\ l^{+} = 2^{-1/2} \, (t - x) \end{cases} \qquad \begin{cases} t = 2^{-1/2} \, (l^{-} + l^{+}) \\ x = 2^{-1/2} \, (l^{-} - l^{+}) \end{cases} \tag{2}$$

$$g = 2 \, dl^{-} \, dl^{+} \tag{3}$$

By conformal transformations we can transform to any orthogonal system. The new coordinates are denoted by a high case letter to contrast with the old ones:

$$\Omega^2 \; dL^- \; dL^+ \; = \; dl^- \; dl^+, \qquad 4$$

(T, X) are related to the L's as in equation 2. There is no restriction in taking

$$\Omega^2 = \frac{dl^-}{dL^-} \; \frac{dl^+}{dL^+} \qquad 5$$

The d'Alembertian becomes

$$\Box_\eta = \partial_t^2 - \partial_x^2 = 2\,\partial_- \; \partial_+ \implies \Box_g = \Omega^{-2} \; \Box_\eta \qquad 6$$

and the new Klein-Gordon equation is

$$(\partial_T^2 - \partial_X^2) \; U(T,X) = -m^2 \; \Omega^2 \; U(T,X) \qquad 7$$

This holds in any system of coordinates that satisfies orthogonality.

We now ask for separability and write

$$\Omega^2 = Y^2(T) - Z^2(X) \qquad 8$$

where Y and Z are complex functions of one variable. That means

$$\partial_T \; \partial_X \; \Omega^2 = 0 \qquad 9$$

and from

$$2\,\partial_T\,\partial_X = \partial_-^2 - \partial_+^2 \qquad 10$$

we have

$$\frac{dL^-}{dl^-}\,\frac{d^3 l^-}{dL^{-3}} = \frac{dL^+}{dl^+}\,\frac{d^3 l^+}{dL^{+3}} = \nu \qquad 11$$

These differential equations define the orthogonal separable coordinates. They are hyperbolic for $\nu > 0$, parabolic for $\nu = 0$ and elliptic for $\nu < 0$.

Hyperbolic coordinates are given by

$$\begin{cases} t + x = \frac{1}{w}\,[\,d_1\,e^{-w\,(T+X)} + d_2\,e^{w\,(T+X)}\,] \\ t - x = \frac{1}{w}\,[\,d_3\,e^{-w\,(T-X)} + d_4\,e^{w\,(T-X)}\,] \end{cases} \qquad 12$$

with

$$d_i = \mathrm{sgn}\ d_i \qquad \text{and } \nu = 2\,w^2 . \qquad 13$$

The conformal factor is

$$\Omega^2 = d_1\,d_3\,e^{-2wT} + d_2\,d_4\,e^{2wT} - d_1\,d_4\,e^{-2wX} - d_2\,d_3\,e^{2wX} . \qquad 14$$

Parabolic coordinates are given by

$$\begin{cases} l^- = d_1\,L^- + \frac{d_2}{2}\,(L^-)^2 \\ l^+ = d_3\,L^+ + \frac{d_4}{2}\,(L^+)^2 \end{cases} \qquad 15$$

with conformal factor

$$\Omega^2 = d_1 d_3 + \frac{d_1d_4 + d_2d_3}{\sqrt{2}} T - \frac{d_1d_4 - d_2d_3}{\sqrt{2}} X + \frac{d_2d_4}{2} (T^2 - X^2) . \qquad 16$$

Elliptic coordinates are given by

$$\begin{cases} t + x = \frac{1}{w} [d_1 \cos w(T + X) + d_2 \sin w(T + X)] \\ t - x = \frac{1}{w} [d_3 \cos w(T - X) + d_4 \sin w(T - X)] \end{cases} \qquad 17$$

with

$$\nu = - w^2 , \qquad 18$$

The conformal factor is

$$\Omega^2 = \frac{1}{2} [(d_1d_3 + d_2d_4) \cos 2wT - (d_1d_4 + d_2d_3) \sin 2wT +$$

$$+ (d_2d_4 - d_1d_3) \cos 2wX + (d_1d_4 - d_2d_3) \sin 2wX] . \qquad 19$$

We could of course have asked only for flatness and separability and get to equation 8 without the use of an explicit coordinate transformation. We are neverthless looking for a coordinate transformation, so this will be of no advantage.

So let us look at the coordinates. Figures 1 to 12 show the systems of coordinates. It is interesting to note a few properties of these coordinates. System A is the only one that covers the whole Minkowski-space; it is cartesianlike in a neighbourhood of the origin; the integral of the acceleration of a point that follows the timelike coordinate line is finite, much like a boost[21]. Systems B, D and G have one coordinate singularity at $I^+ = 0$; it is an event-horizon for observers going with the timelike coordinate line; for the last two systems the coordinate line touches the horizon needing for that an infinite amount of acceleration in a finite time t; in system B the observers are initially inertial and are smoothly accelerating, so that they become of Rindler type asymptotically. Systems C (the well known Rindler coordinates[22]), E, F and H have two event-horizons at constant l^+ and l^-; in system E the horizons are of two different types. The last system I covers only a compact region of Minkowski-space; it is a very pleasant figure to look at, where one can most easily see one of the important properties of the orthogonal separable coordinates, that is the fact that the coordinate lines are in each case confocal lines in the Minkowski sense[18].

Other informations about the separable orthogonal systems of coordinates can be seen from Tables 1 and 2[11]. The acceleration a at the curve X = const., is given by:

$$a = \frac{1}{2\,\Omega^3}\,\frac{\partial(\Omega^2)}{\partial X} = \frac{Z\,Z'}{(Y^2 + Z^2)^{3/2}} ; \qquad 20$$

A is the proper-time integration of the acceleration:

$$A = Z\;Z' \int_{H^-}^{H^+} \frac{d\,T}{Y^2 + Z^2} ; \qquad 21$$

θ and φ are the angles necessary to rotate the hyperbole so that its axis coincides with the coordinate axis and the angle between its asymptote and these axis:

For the curves with T = constant:

$$\mathrm{tg}\,\theta = \frac{d_1 d_2 - d_3 d_4 + \sqrt{(d_1^2 e^{-2wT} + d_4^2 e^{2wT})(d_3^2 e^{-2wT} + d_2^2 e^{2wT})}}{d_1 d_3 e^{-2wT} + d_2 d_4 e^{2wT}}, \qquad 22$$

$$\mathrm{tg}\,\varphi = \pm \frac{d_1 d_2 + d_3 d_4 + \sqrt{(d_1^2 e^{-2wT} + d_4^2 e^{2wT})(d_3^2 e^{-2wT} + d_2^2 e^{2wT})}}{d_1 d_3 e^{-2wT} - d_2 d_4 e^{2wT}}. \qquad 23$$

For the curves with X = constant:

$$\mathrm{tg}\,\theta = \frac{d_1 d_2 - d_3 d_4 + \sqrt{(d_1^2 e^{-2wX} + d_3^2 e^{2wX})(d_4^2 e^{-2wX} + d_2^2 e^{2wX})}}{d_1 d_4 e^{-2wX} + d_2 d_3 e^{2wX}}, \qquad 24$$

$$\mathrm{tg}\,\varphi = \pm \frac{d_1 d_2 + d_3 d_4 + \sqrt{(d_1^2 e^{-2wX} + d_3^2 e^{2wX})(d_4^2 e^{-2wX} + d_2^2 e^{2wX})}}{d_1 d_4 e^{-2wX} - d_2 d_3 e^{2wX}}. \qquad 25$$

III - Two new vacua

We now proceed to construct quantum field theory in the special case of coordinate system B. We choose it for three reasons: its well known asymptotes; its technical simplicity; and because it gives a unique opportunity to exactly study a transition from inertial to accelerated movement, which alows for the interpretation of particle creation in the framework of only one system of observers. This has an obvious episthemological superiority to the Rindler system. The straightforward application of the recipe given here to other systems will be developed in a forthcoming paper.

Our reasoning here is perhaps naive but we follow the criticism of Hinton[16] about the apparent sureness of more ellaborated approaches; at least, we are simpler. Like Fulling[2] we merely calculate the solutions of the Klein-Gordon equation in two systems of coordinates with the special boundary conditions that we call quasi-classical. After normalizing them, we calculate the Bogoliubov-coefficient, that give the expectation value of the number-operator in this basis over the vacuum of the cartesian plane waves. We use the conventions and notation of Birrel, Davies[23].

We go back to the Klein-Gordon equation and choose a basis of solutions to build up the Fock space. Equations 7 and 8 imply

$$\begin{cases} F'' + (m^2 Y^2 + K^2)\, F = 0 \\ G'' + (m^2 Z^2 + K^2)\, G = 0, \end{cases} \qquad 26$$

with

$$U(T,X) = F(T) \cdot G(X). \qquad 27$$

To state the orthonormality conditions

$$\langle U_K \mid U_L \rangle = \delta(K - L), \text{ etc.} \tag{28}$$

we define the scalar product as

$$\langle U , V \rangle := - i \int_S dS^a \; U \overleftrightarrow{\partial}_a V^* \tag{29}$$

where S^a is a Cauchy-surface and dS^a is a future pointing unit vector:

$$dS^a = (g)^{1/2} g^{ab} \epsilon_{bc\ldots n} dx^c \wedge \ldots \wedge dx^n. \tag{30}$$

Now we choose the Cauchy-surface so that it is a coordinate line. This is possible for all of our coordinate systems, as you can see in the figures. We call "ext" the value of T where this happens, "min", "max" the extrema of X:

$$dS^a = \eta^{ab} \epsilon_{bc} X^c_, \tag{31}$$

with $Y = Y_{ext}$ we have $dY = 0$ and

$$dS^a = (dZ \, \partial X / \partial Z, 0) \tag{32}$$

so that

$$\langle U \mid \bar{U} \rangle = - i \; F \overleftrightarrow{d}_T \bar{F}^* \int dX \; G \bar{G}^* . \tag{33}$$

The choosen surface is in general not a Cauchy-surface but the boundary values are such that this is indeed afforded. This is a standard procedure but it is not unique as noted by Rumpf[24]. Equation 33 implies

$$\langle U_K \mid U_L \rangle = \left. \frac{- i \; F_K \overleftrightarrow{d}_T F^*_L}{K^2 - L^{2*}} \right|_{T = ext} \left. G_K \overleftrightarrow{d}_X G^*_L \right|_{X = min}^{max} . \tag{34}$$

We will say that a given mode is natural in a domain if it is quasi-classical there. This means that

$$S \gg A \tag{35}$$

in the Ansatz

$$U = A \exp(iS). \qquad 36$$

In this limit the Klein-Gordon equation transforms into the Hamilton-Jacobi equation:

$$S_{;a} S^{;a} = m^2 , \qquad 37$$

so that in the conformal metric

$$\left(\frac{\partial S}{\partial T} \right)^2 - \left(\frac{\partial S}{\partial X} \right)^2 = \Omega^2 m^2 , \qquad 38$$

and

$$S(T,X) = \pm \int dT \sqrt{K^2 + m^2 Y^2(T)} \pm \int dX \sqrt{K^2 + m^2 Z^2(X)} \qquad 39$$

gives the asymptote of the quasi-classical solution.

We illustrate this in the special case of the coordinates B, that is,

$$\begin{cases} t + x = 2/w \sinh (w (T + X)) \\ t - x = - 2/w \exp (-w (T - X)). \end{cases} \qquad 40$$

With the new variables

$$\begin{cases} Y = \left| \frac{-e^{-wT}}{w} \right| = \frac{e^{-wT}}{w} \\ Z = \left| \frac{ie^{-wX}}{w} \right| = \frac{e^{wX}}{w} \end{cases} \qquad 41$$

equations 26 are two Bessel equations with imaginary order $i\nu$, where we write

$$\nu = K / w \qquad 42$$

for simplicity. The quasi-classical asymptotes used as boundary conditions select two orthonormal solutions:

$$U_K^{in} = \frac{\sqrt{K\,(e^{2\pi\nu} - 1)\,/\,\pi}}{2\,w} \quad H^2_{i\nu}\,(mY)\ \ K_{i\nu}\,(mZ), \qquad 43$$

$$U_K^{out} = \frac{1}{w}\sqrt{\frac{K}{\pi}}\ \ J_{-i\nu}\,(mY)\ K_{i\nu}\,(mZ). \qquad 44$$

They are quasi-classical at $Y \to \infty$ or $\to 0$ and $Z \to \infty$ respectively. For details, see reference11.

We may in general compare the two bases U_K and u_k through the Bogoliubov coefficients A and B:

$$U_K = A_{Kk}\,u_k + B_{Kk}\,u_k^{*}. \qquad 45$$

The vacuum states defined with the help of annihilation operators C_K and c_k are also related by means of these coefficients. In particular the expectation value of the number operator of k-particles in the K-vacuum is given by:

$$\langle\, 0_K \mid N_k \mid 0_K \,\rangle = \int dK\ \mid B_{Kk} \mid^2, \qquad 46$$

We now compare the two bases U^{in} and U^{out}. The Bogoliubov coefficient

$$B^{o\,i}_{K\,L} = -\,\langle\, U_K^{out}\,,\,U_L^{in\,*}\,\rangle \qquad 47$$

is, much like equation 34,

$$B^{o\,i}_{K\,L} = \frac{i}{K^2 - L^2}\,\left(\frac{1}{w}\sqrt{\frac{K}{\pi}}\ \frac{\sqrt{K(e^{2\pi\nu}-1)/\pi}}{2w}\right)\left[\,J_{-i\nu}\,wY\overleftrightarrow{\partial}_Y\,H^1_{i\mu}\,\right|_{\infty} \qquad 48$$

$$\cdot\quad \left[\,K_{i\nu}\,wZ\overleftrightarrow{\partial}_Z\,K_{i\mu}\,\right|_0^{\infty},$$

so that

$$B^{o\,i}_{K\,L} = \frac{1}{\sqrt{e^{2\pi K/w} - 1}} \; \delta\,(K - L). \qquad 49$$

For these calculations we refer to chapter 3 of Magnus, Oberhettinger, Soni[25].

Next we compare these bases with the cartesian one, that is, with plane waves in the original coordinates,

$$u_k = 1/2\ (\pi\, e)^{-1/2}\ \exp\,(-\,i\,(e\,t - k\,x)). \qquad 50$$

In the variables (Y, Z), it holds

$$\begin{cases} e\,t - k\,x = -\dfrac{1}{2}\left[\left(\dfrac{e-k}{w}\right)\left(\dfrac{Y}{Z} - \dfrac{Z}{Y}\right) + w\,(e+k)\,Y\,Z \right] \\ \partial_T\, u_k = -\dfrac{i}{2}\left[(e-k)\left(\dfrac{Y}{Z} + \dfrac{Z}{Y}\right) + w^2 (e+k)\,Y\,Z \right] u_k, \end{cases} \qquad 51$$

so that their limit when $Y \to 0$ is

$$\begin{cases} e\,t - k\,x = \dfrac{e-k}{2\,w\,Y}\; Z \\ \partial_T\, u_k = -\dfrac{i}{2}\ \dfrac{e-k}{Y}\ Z\, u_k. \end{cases} \qquad 52$$

The Bogoliubov coefficient

$$B_{Kk} = -\,\langle\, U_K \mid u^*_k \,\rangle \qquad 53$$

is a sum of two integrals:

$$I_1 := \int dX\ G_K\,(X)\ u_k\,(t\,(T,X)\,,\ x\,(T,X)\,) \qquad 54$$

and

$$I_0 := \int dX\, B_K(X)\, \partial_T\, u_K\, (t\,(T, X)\,,\, x\,(T, X)\,). \qquad 55$$

If we put in equation 55 the expressions 50 and 52 we get the Fourier transformation

$$I_0 = \frac{1}{2\sqrt{\pi e}} \left[-i \left(\frac{e-k}{2Y}\right) \right] \int \frac{dZ}{w}\, K_{i\nu}(mZ)\, \exp\left[-i \left(\frac{e-k}{2wY}\right) Z \right]. \qquad 56$$

I_1 is the integral of I_0 in the variable $-i\,(e-K)\,/\,2wY$. These two integrals are easy to evaluate in the Limit $Y \to 0$[26]. After the inclusion of F_K we get

$$B^{out\ cart}_{Kk} = \sqrt{\frac{K}{e}}\ \frac{\exp\,(-\pi\nu/2)}{w\,(-i\nu)!\ 2\sinh \pi\nu}\ \left(\frac{e-k}{2w}\right)^{i\nu} \qquad 57$$

and

$$B^{in\ cart}_{Kk} = -\frac{1}{w}\,(\sinh \pi\nu)^{-3/2}\ \sqrt{\frac{K}{e}}\ \mathrm{Re}\left[\frac{((e-k)/2w)^{-i\nu}}{(-i\nu)!}\right]. \qquad 58$$

Their absolute square is

$$\left|B^{out\ cart}_{Kk}\right|^2 = \frac{1}{2\,\pi\, e\, w}\ \frac{1}{e^{2\pi K/w} - 1} \qquad 59$$

and

$$\left|B^{in\ cart}_{Kk}\right|^2 = (\sinh \pi\nu)^{-3}\ \frac{K}{e\, w^2}\ \mathrm{Re}^2\left[\frac{((e-k)/2w)^{-i\nu}}{(-i\nu)!}\right]. \qquad 60$$

We discuss these results in the following section.

IV - Conclusions

The results of the last section show that the out-vacuum is, compared to the cartesian one, a thermal state with temperature

$$\Theta_0 = w / 2\pi k_B . \qquad 61$$

After Tolmann27 this means that in the proper frame the temperature is

$$\Theta = \Theta_0 \ (g_{00})^{-1/2} \qquad 62$$

or, with

$$\lim_{Y\to 0} g_{00} = \exp(2wX), \qquad 63$$

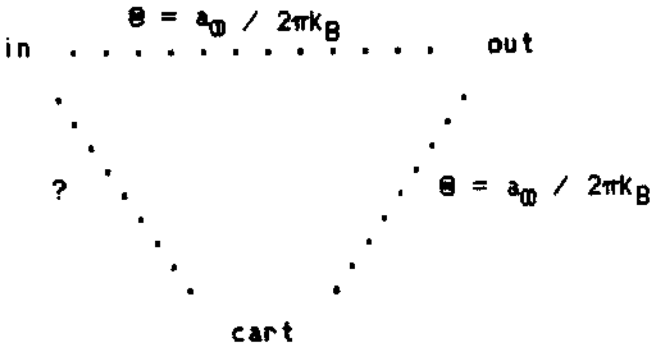
$$\Theta = \frac{w \exp(-wX)}{2 \pi k_B} = \frac{a_\infty}{2 \pi k_B}, \qquad 64$$

where

$$a_\infty = \lim_{Y\to 0} a(T,X) = w \exp(-w X) \qquad 65$$

Also, compared to the in-vacuum, the out-vacuum has the same Fulling temperature. The in-vacuum compared to the cartesian one is not a thermal state. We illustrate this situation in the folowing diagramm.

The "natural" vacua of the system of coordinates B:



The Fulling effect make this result expected. The observer defines its "natural" - in accordance with our boundary conditions - vacuum as long as it is inertial; then it accelerates and reaches a uniform acceleration a_∞, thus seeing a thermal sea of particles around it, with temperature $\Theta = a_\infty / 2\pi k_B$. For this observer the "natural" particle number is not conserved.

The obtained temperatures are in accordance with the result of Sanchez for massless particles[28]. It is not surprising, that the in-vacuum is not a perfect vacuum as compared to the cartesian one: the exactness of the approach forces the in-states to contain vestiges of the out-one.

In a following paper we intend to extend this analysis to the calculation of other physical magnitudes, like $\langle T_{ik} \rangle$. Likewise we will consider with the other separable orthogonal systems of coordinates. We hope then to be able to atack the problem in a more abstract level, guided by the our new knowledge on non-static systems of coordinates. The generalization to 3 dimensions allows the immediate verification of a speculation of Grove, Otewill[9], that says that rigidity is the criterium to choose the "good" detector by handling a system of coordinates that is rigid but non-static.

We want to thank H. Rumpf for suggesting the subject of this work, R. Beig and H. Urbantke for their assistance and the whole Institut fuer theoretische Physik of the University of Vienna for its hospitality during my stay there.

Figure Captations

Figure 1: The system of coordinates A

Drawn are the curves T = n/4 from X = -1 to 1, and X = n/4 from T = -1 to 1, where n = -4, -3, ..., 4. The t-axis is the vertical, the x-axis the other one; the primes go from -1 to 1.

Figure 2: The system of coordinates B

Drawn are the curves T = n/3 from X = -1 to 1, and X = n/3 from T = -1 to 1, where n = -3, -2, ..., 3. The axes are as in figure 1.

Figure 3: The system of coordinates C_R

Drawn are the curves T = n/2 from X = -1 to 1, and X = n/2 from T = -1 to 1, where n = -2, -1, ..., 2. The axes are as in figure 1.

Figure 4: The system of coordinates C_M

Drawn are the curves T = n/2 from X = -1 to 1, and X = n/2 from T = -1 to 1, where n = -2, -1, ..., 2. The axes are as in figure 1.

Figure 5: The system of coordinates D

Drawn are the curves T = n/3 from X = -1 to 1, or X = n/3 from T = -1 to 1, where n = -3, -2, ..., 3. The axes are as in figure 1.

Figure 6: The system of coordinates E_+

Drawn are the curves T = n/3 from X = -1 to 1, or X = n/3 from T = -1 to 1, where n = -3, -2, ..., 3. The axes are as in figure 1.

Figure 7: The system of coordinates E_-

Drawn are the curves T = n/3 from X = -1 to 1, or X = n/3 from T = -1 to 1, where n = -3, -2, ..., 3. The axes are as in figure 1.

Figure 8: The system of coordinates F_+

Drawn are the curves T = n/4 from X = -1 to 1, or X = n/4 from T = -1 to 1, where n = 0, 1, ..., 4. The axes are as in figure 1.

Figure 9: The system of coordinates F_-

Drawn are the curves $T = n/4$ from $X = -1$ to 1, or $X = n/4$ from $T = -1$ to 1, where $n = 0, 1, \ldots, 4$. The axes are as in figure 1.

Figure 10: The system of coordinates G

Drawn are the curves $T = n$ from $X = -3$ to 3, or $X = n$ from $T = -1$ to 1, where $n = -3, -2, \ldots, 3$. The t-axis is the vertical, the x-axis the other one; the primes go from -3 to 3.

Figure 11: The system of coordinates H_+

Drawn are the curves $T = n/2$ from $X = -3$ to 3, or $X = n/2$ from $T = -3$ to 3, where $n = 0, 1, 2, 3$. The axes are as in figure 10.

Figure 12: The system of coordinates H_-

Drawn are the curves $T = n/2$ from $X = 0$ to 3, or $X = n/2$ from $T = 0$ to 3, where $n = -3, -2, \ldots, 3$. The axes are as in figure 10.

Figure 13: The system of coordinates I

Drawn are the curves $T = n\pi/6$ from $X = -\pi$ to π, or $X = n\pi/6$ from $T = -\pi$ to π, where $n = 0, 1, \ldots, 3$. The t-axis is the vertical, the x-axis the other one; primes go from -1 to 1.

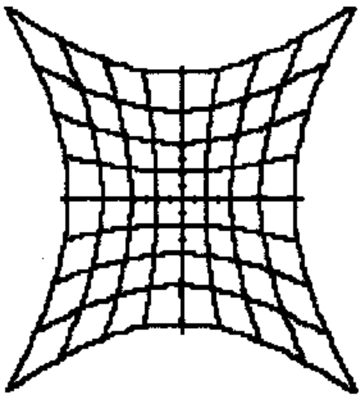


Figure 1

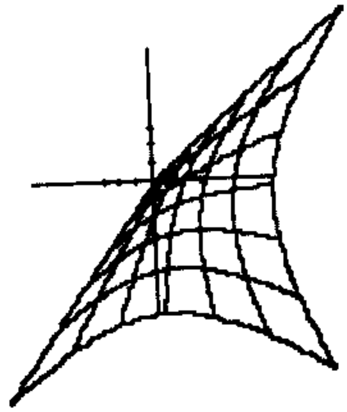


Figure 2

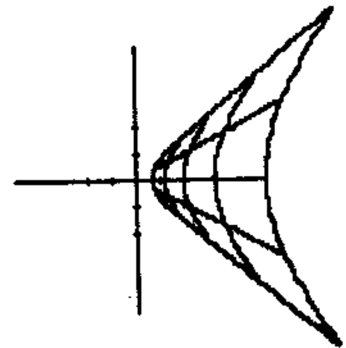


Figure 3

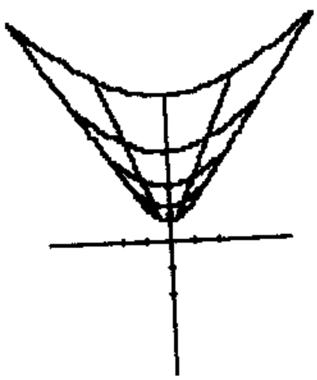


Figure 4

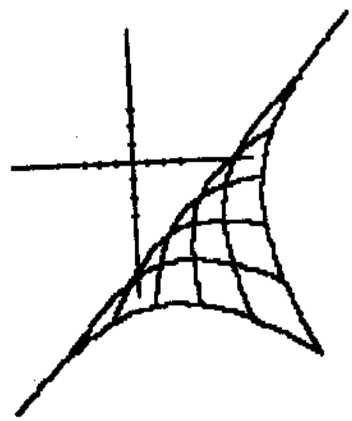


Figure 5

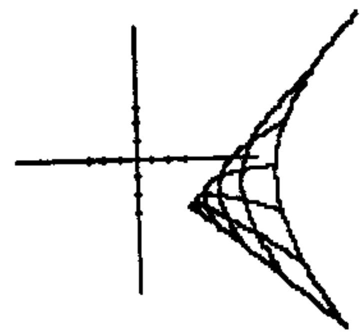


Figure 6

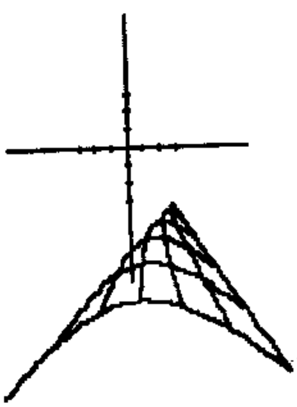


Figure 7

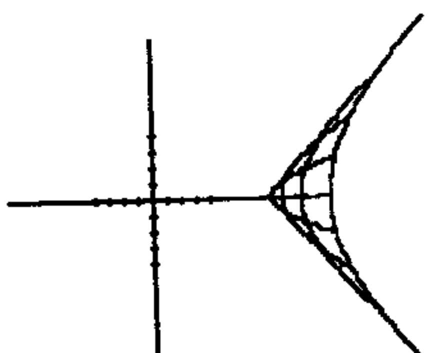


Figure 8

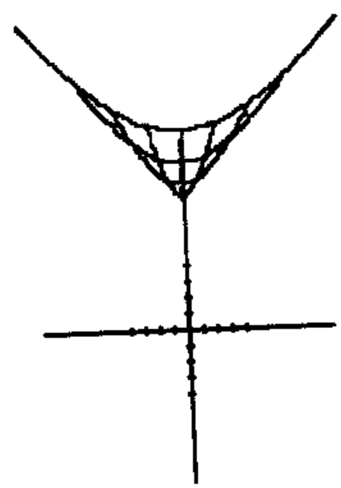


Figure 9

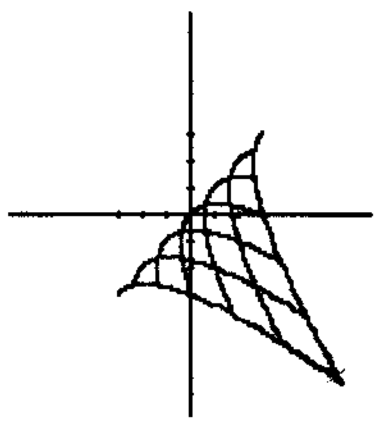


Figure 10

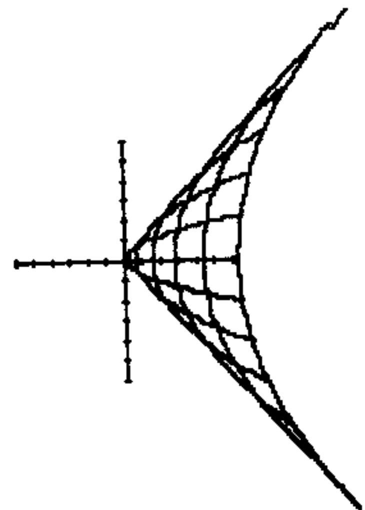


Figure 11

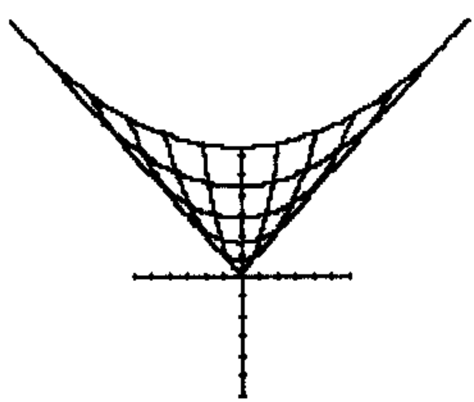


Figure 12

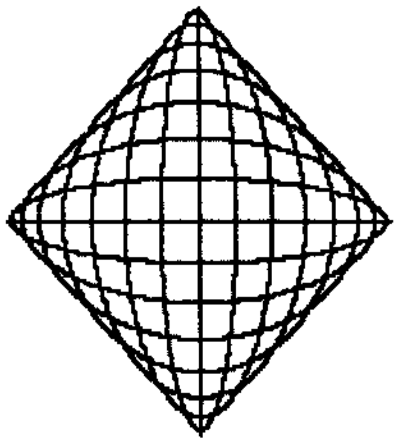


Figure 13

Table 1: The hyperbolic systems of coordinates

SC	Conformal factor	d	Transformation or curve parameters θ, φ	Comments
A	$2(\mathrm{ch}2wT + \mathrm{ch}2wX)$	-1 1 -1 1	$t = 2/w\ \mathrm{sh}\ wT\ \mathrm{ch}\ wX$ $x = 2/w\ \mathrm{ch}\ wT\ \mathrm{sh}\ wX$	$A = 2wX$
B	$e^{-2wT} + e^{2wX}$	-1 1 -1 0	$\mathrm{tg}\ 2\theta = \mathrm{tg}\ 2\varphi = -e^{-2wT}$ $\mathrm{tg}\ 2\theta = \mathrm{tg}\ 2\varphi = e^{2wX}$	$a = \dfrac{w\,e^{2wX}}{(e^{-2wT} + e^{2wX})^{3/2}}$
C_R	e^{2wX}	0 1 -1 0	$t = \frac{1}{w} e^{wX}\ \mathrm{sh}\ wT$ $x = \frac{1}{w} e^{wT}\ \mathrm{ch}\ wT$	$a = w\,e^{-wX}$, $A = wT$
C_M	e^{2wT}	0 1 0 1	$t = \frac{1}{w} e^{wT}\ \mathrm{ch}\ wX$ $x = \frac{1}{w} e^{wT}\ \mathrm{sh}\ wX$	$a = 0$
D	$2(\mathrm{sh}2wX - \mathrm{sh}2wT)$	-1 1 -1 -1	$\mathrm{tg}\ \theta = -\mathrm{tgh}wT$, $\mathrm{tg}\varphi = -\pm 1$ $\mathrm{tg}\ \theta = -\mathrm{ctgh}wX$, $\mathrm{tg}\varphi = -\pm 1$	
E_+	$e^{2wT} - e^{2wX}$	0 1 -1 -1	$\mathrm{tg}\ 2\theta = -e^{2wT}$ $\mathrm{tg}\ 2\varphi = \mathrm{tg}\ 2\theta$	$a = \dfrac{-w\,e^{2wX}}{(e^{2wT} - e^{2wX})^{3/2}}$
E_-	$e^{2wX} - e^{2wT}$	0 -1 -1 -1	$\mathrm{tg}\ 2\theta = e^{2wT}$ $\mathrm{tg}\ 2\varphi = -\ \mathrm{tg}\ 2\theta$	$a = \dfrac{w\,e^{2wX}}{(e^{2wX} - e^{2wT})^{3/2}}$
F_+	$2(\mathrm{ch}2wT - \mathrm{ch}2wX)$	1 1 1 1	$t = 2/w\ \mathrm{ch}\ wT\ \mathrm{ch}\ wX$ $x = 2/w\ \mathrm{sh}\ wT\ \mathrm{sh}\ wX$	
F_-	$2(\mathrm{ch}2wX - \mathrm{ch}2wT)$	1 1 -1 -1	$t = 2/w\ \mathrm{sh}\ wT\ \mathrm{sh}\ wX$ $x = 2/w\ \mathrm{ch}\ wT\ \mathrm{ch}\ wX$	

Table 2: The parabolic and elliptic systems of coordinates

SC	Conformal factor	d	Transformation of coordinates	Acceleration
G	$\frac{2}{d}(T+X)$	$0 \quad \frac{\sqrt{8}}{d}$ $1 \quad 0$	$l^{-} = (T+X)^2 / \sqrt{2}\, d$ $l^{+} = (T-X) / \sqrt{2}^{-}$	$\sqrt{\frac{d}{8}}\,(T+X)^{-3/2}$
H_+	$\frac{2}{d^2}(T^2 - X^2)$	$0 \quad \frac{\sqrt{8}}{d}$ $0 \quad \frac{\sqrt{8}}{d}$	$t = \frac{1}{d}(T^2 + X^2)$ $x = \frac{2}{d}\, T\, X$	$-\frac{d}{2}\,\frac{X}{(T^2 - X^2)^{3/2}}$
H_-	$\frac{-2}{d^2}(T^2 - X^2)$	$0 \quad \frac{\sqrt{8}}{d}$ $0 \quad \frac{-\sqrt{8}}{d}$	$t = \frac{2}{d}\, T\, X$ $x = \frac{1}{d}(T^2 + X^2)$	$-\frac{d}{2}\,\frac{X}{(X^2 - T^2)^{3/2}}$
I	$\frac{(\cos 2wT - \cos 2wX)}{2}$	$1 \quad 0$ $1 \quad 0$	$t = -\frac{1}{w}\sin wT \sin wX$ $x = \frac{1}{w}\cos wT \cos wX$	$\sqrt{2w}\,\frac{\sin 2wX}{(\cos 2wT - \cos 2wX)^{3/2}}$

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