ISSN 0029-3865

CBPF-NF-024/86

AN UNUSUAL ROAD TO CHAOS

by

M.C. de Sousa Vieira, E. Lazo* and C. Tsallis

Centro Brasileiro de Pesquisas Físicas - CNPq/CBPF
Rua Dr. Xavier Sigaud, 150 - Urca
22290 - Rio de Janeiro, RJ - Brasil

*Permanent address: Departamento de Física, Facultad de Ciencias, Universidad de Tarapacá, Arica, Chile.

ABSTRACT

We numerically discuss the assymmetric map $x'=1-\varepsilon_i-a_i|x|^{z_i}$ (i=1,2 respectively correspond to $x>0$ and $x<0$). Severe differences appear with respect to the Feigenbaum scenario ($\varepsilon_1=\varepsilon_2=0$; $a_1=a_2$; $z_1=z_2$), the strongest corresponding to simple discontinuity ($\varepsilon_1\neq\varepsilon_2$; $a_1=a_2$; $z_1=z_2$) in which case many inverse cascades are observed. The whole set of these cascades can be seen as a new road to chaos.

Key-words: Chaos; One-dimensional maps; Assymmetric maps; Universality class.

Since the pionner work of May[1], the maps on the interval (one-dimensional dissipative maps) have been object of increasing interest, both due to their intrinsic mathematical richness and to the large number of physical systems[2] which experimentally display transitions into chaos via Feigenbaum bifurcations. One of the most studied maps is the following one (see [3] and references therein):

$$x_{t+1} = 1 - a|x_t|^z \qquad (z \geq 1) \tag{1}$$

For $z=2$, it is equivalent to the standard logistic map ($x_{t+1} = 4\mu x_t(1-x_t)$). When a increases from 0 to $a^*(z)$ ($a^*(2) = 1.401155...$; see [3] for $a^*(z)$), the attractor (or long-time solution) of the map (1) exhibits a sequence of periodic orbits with periods 2^k ($k=0,1,2,...$); the k-th period appears at a_k through the pitchfork bifurcation of the (k-1)-th period; the sequence $\{a_k\}$ accumulates ($k \to \infty$) at $a^*(z)$, where the system enters into chaos. For $a > a^*(z)$, an infinite number of p-furcation "windows" ($p=2,3,4...$) occur (in a non trivial order), up to $a^M(z)$ ($a^M(z) = 2$ for $z \geq 1$) above which no finite attractor persists, and x_t is driven to infinity. At the precise value $a = a^M(z)$, the map essentially becomes a generator of random real numbers, as the density of the successive x_t tends towards a distribution which is equivalent to a completely "flat" distribution. With each p-furcation window we can associate[3,4], for fixed z, a critical exponent $\delta(z)$ (as well as other critical exponents, e.g., $\alpha(z)$) by considering the location of the successive p-furcations. For example, for the simple bifurcation series which accumulates on $a^*(z)$, we can define $\delta_k \equiv (a_k - a_{k-1})/(a_{k+1} - a_k)$, and verify that $\delta \equiv \lim_{k\to\infty} \delta_k$ is a finite num ber ($\delta(2) = 4.669...$; see [3] for $\delta(z)$). Each value of $\delta(z)$ determines an *universality class* in the sense that it is shared, for that particular window, by (almost) all one-dimensional maps presenting a single extremum of the $|x|^z$-type.

The aim of the present paper is to numerically study the influence, on the above scenario, of *assymmetry* in the extremum, i.e., different $x \to +0$ and $x \to -0$ behaviours: this type of assymmetry seems to appear in physical systems[1,5]. To do this study we shall generalize the map (1), hereafter referred to as the *prototype map*, into

$$x_{t+1} = f(x_t) \equiv \begin{cases} 1 - \varepsilon_1 - a_1|x_t|^{z_1} & \text{if} \quad x_t > 0 \ , \\ 1 - \frac{1}{2}\,(\varepsilon_1+\varepsilon_2) & \text{if} \quad x_t = 0 \ , \\ 1 - \varepsilon_2 - a_2|x_t|^{z_2} & \text{if} \quad x_t < 0 \ , \end{cases} \tag{2}$$

with z_1, $z_2 \geq 1$ (the prototype is obviously recovered for $\varepsilon_1 = \varepsilon_2 = 0$, $a_1 = a_2 \equiv a$, and $z_1 = z_2 \equiv z$). This map yields, for fixed z_1 and z_2, a sorte of phase diagram in the $(\varepsilon_1, \varepsilon_2, a_1, a_2)$-space, in the sense that complex sets of p-furcation hypersurfaces can be defined therein which eventually accumulate on special hypersurfaces. One important such hypersurface corresponds to the first entrance into chaos (generalization of $a^*(z)$); another one corresponds to the disappearance of finite attractors (generalization of $a^M(z)$). Another interesting information concerns the evolution of the attractor (set of values of x towards which x_t tends in the $t \to \infty$ limit) as a function of $(\varepsilon_1, \varepsilon_2, a_1, a_2)$. Finally, the Liapunov exponent λ is an important quantity to be known (as a function of $(\varepsilon_1, \varepsilon_2, a_1, a_2)$), as it characterizes the sensitivity to initial conditions ($\lambda < 0$ corresponds to periodic orbits, and $\lambda > 0$ corresponds to chaotic motion; λ vanishes on every pitchfork bifurcation, and, in particular, on the "first entrance into chaos" - hypersurface mentioned above). The exponent λ is defined through

$$\lambda(z_1,z_2) \equiv \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} \ell n \left| \frac{df(x)}{dx} \right|_{x=x_t} \quad (3)$$

To analyze the main consequences, on the "first entrance into chaos" and "finite attractor disappearance" hypersurfaces as well as on the evolution of the attractor and of λ, of the asymmetry introduced in the map, we shall proceed by modifying one type of parameters (the ε's, the a's and the z's) at a time. We shall therefore consider three cases, namely case I ($\varepsilon_1,\varepsilon_2 \neq 0$, $a_1=a_2 \equiv a$, and $z_1=z_2 \equiv z$), case II ($\varepsilon_1=\varepsilon_2=0$, $a_1 \neq a_2$, and $z_1=z_2 \equiv z$), and case III ($\varepsilon_1=\varepsilon_2=0$, $a_1=a_2 \equiv a$, and $z_1 \neq z_2$).

Case I: We have represented in Fig. 1(a), for $z_1=z_2=2$ and $a_1=a_2=a$, the $(\varepsilon_1,\varepsilon_2)$-dependence of a^* and a^M: note that a^* and a^M merge for ε_1 and ε_2 high enough. It is also worthy to mention that: (i) for $\varepsilon_1 \neq \varepsilon_2$, a^M depends on z, contrarily to what happens for $\varepsilon_1=\varepsilon_2$, where a^M is independent from z; (ii) for $\varepsilon_1 \neq \varepsilon_2$, the attractor attains $|x| > 1$ for a's slightly smaller than those which make the finite attractor disappear, contrarily to what happens for $\varepsilon_1=\varepsilon_2$, where the two phenomena occur *simultaneously*.

In Fig. 2 we have represented, for $z_1=z_2=2$ and $a_1=a_2 \equiv a$, the a-dependence of the attractor and of λ. Two typical cases have been illustrated, namely $(\varepsilon_1,\varepsilon_2) = (0,0.1)$ and $(0.1,0)$. In both cases a remarkable feature is observed: the appearance of windows of inverse p-furcations. These p-furcations appear discontinuously like tangent bifurcations, *but do not present intermittency* (these facts become intelligible if the iterated functions f(f(...f(x))), are represented, as square corners approach and cross the $x'=x$ bissectrix). A big (maybe infinite) number of such windows appear. Within each of these windows the p-furcations accumulate for *decreasing* a, but *no chaos appears in the neighborhood of the accumulation points*. For example, for $(\varepsilon_1,\varepsilon_2) = (0,0.1)$, the first of such cascades appears immediately above a=1. The sequence of periods (cycle size) is as follows: $...16 \to 14 \to 12 \to 10 \to 8 \to 6 \to 4$, and they accumulate on a=1. Immediately above this cascade we observe a couple of standard pitchfork bifurcations. Further on a new inverse cascade appears as follows: $...33 \to 29 \to 25 \to 21 \to 17 \to 13 \to 9$, and then again a standard pitchfork bifurcation into a period 18. Then another inverse cascade as follows: $...76 \to 58 \to 40 \to 22$. After this cascade, no other standard pitchfork bifurcations are observed, but instead more inverse cascades: $...92 \to 70 \to 48 \to 26$, and then $...134 \to 108 \to 82 \to 56$, and then $...198 \to 142 \to 86 \to 30$, and then $...124 \to 94 \to 64 \to 34$, etc. Two simple rules emerge: (i) when standard pitchfork bifurcations no more appear, most windows have inverse cascades whose first element equals a previous first element plus 4; (ii) within each window, the periods grow *arithmetically* by adding the first element of the previous window. Inverse cascades mixtured within the Feigenbaum scenario have already been registered in the literature[6] for Hamiltonian systems, but while those exhibit *few* windows with *continuous* bifurcations whose periods increase *geometrically*, the present ones exhibit *many* windows with *discontinuous* p-furcations whose windows increase *arithmetically*. In the $(\varepsilon_1,\varepsilon_2) = (0.1,0)$ case, after a couple of standard pitchfork bifurcations (including, however, an unusual jump), an inverse cascade appears as follows: $...22 \to 18 \to 14 \to 10$, and then $...46 \to 36 \to 26$, and then $...120 \to 94 \to 68$, and then $...246 \to 178 \to 110$, and then (new suprise!) the cascade reverses and becomes *direct*, now yealding $110 \to 152 \to 194 \ldots$, and then the period 42 appears (which precisely is the arithmetic step of the direct cascade). All these features are quite amazing on the whole, and give strong evidence of the important role that discontinuities in the maps might have. Furthermore they are *possibly not too hard to exhibit experimentally* (e.g., see in Fig. 2(e) the size of the attractor of period 6). Finally, note in Fig. 2(1) another uncommon fact, namely a discontinuity in λ.

Case II: We have represented in Fig. 1(b), for $z_1 = z_2 = 2$ and $\varepsilon_1 = \varepsilon_2 \equiv 0$, the (a_1, a_2) - critical lines which generalize a^* (first entrance into chaos) and a^M (finite attractor disappearance). On the whole, the Feigenbaum scenario is preserved (with soft translation-like deformations); in particular, the set $\{\delta_k\}$ approaches the Feigenbaum value 4.669.... Summarizing, no important new features are observed in this case.

Case III: In Fig. 2 we have represented, for $\varepsilon_1 = \varepsilon_2 = 0$ and $z_1 \neq z_2$, the a-dependence of the attractor and of λ. Two typical cases have been discussed namely $(z_1, z_2) = (2,4)$ and $(4,2)$. The former yields $a^* \simeq 1.6414$, the latter yields $a^* \simeq 1.3617$, and both yield $a^M = 2$. Below a^*, the sequence of bifurcations is the same as that of Feigenbaum, but a strongly different behaviour appears, as already noticed[7], in the set $\{\delta_k\}$: see Fig. 3 (convenient way for presenting our results as well as the numerical ones obtained in Ref. [7]). Above a^* (chaotic region), the relative sizes of the various windows are quite different from those of the $z_1 = z_2$ prototype. A further analysis would be needed to verify if the prototype sequence of high-order windows is preserved.

We have verified that the Feigenbaum scenario for one-dimensional one-extremum maps is strongly modified if assymmetry is introduced in the extremum. Amplitude assymmetry ($a_1 \neq a_2$) has a relatively minor influence. Exponent assymmetry ($z_1 \neq z_2$) does not seem to drastically alter the bifurcations sequence, but introduces remarkable numerical differences while approaching the first entrance to chaos: the unique geometrical tendency associated with the set $\{\delta_k\}$ disappears. But no doubt it is discontinuity at the extremum ($\varepsilon_1 \neq \varepsilon_2$) which introduces by far the strongest perturbation into the system: the Feigenbaum pitchfork bifurcation scheme quickly disappears (while increasing a), and a possibly infinite number of *inverse* cascades of p-furcations takes place (sometimes mixtured with direct cascades). The p-furcations appear discontinuously (like tangent bifurcations) but do not exhibit intermittency (unlike tangent bifurcations). Within each cascade the p-furcations accumulate on points in the neighborhood of which no chaos is present (negative Liapunov exponent λ), in contrast with the Feigenbaum road. Finally, the size of the periods grows, while approaching the accumulation point, *arithmetically* and not *geometrically* as in the usual case. We are presently working to see whether this complex picture remains somehow similar to itself within smaller scales while approaching a* (birth of positive λ's). If this is the case, then universality properties will emerge, and consequently the present scenario could be in some sense looked at as a new road to chaos.

CAPTION FOR FIGURES

Fig. 1 - Special cuts of the "first entrance into chaos" and the "finite attractor disappearance" hypersurfaces in the $(\varepsilon_1, \varepsilon_2, a_1, a_2)$ - space for $z_1 = z_2 = 2$: (a) $a_1 = a_2 \equiv a$; (b) $\varepsilon_1 = \varepsilon_2 \equiv 0$. For $\varepsilon = 0$ and $a_1 = a_2$, it is $a^* = 1.401155, \ldots$, and $a^M = 2$. We have used the initial point $x_0 = 0.5$.

Fig. 2 - Influence of ε_1, ε_2, a_1, a_2, z_1 and z_2 on the attractor set (x) and the Liapunov exponent (λ). Prototype: $\varepsilon_1 = \varepsilon_2 = 0$, $a_1 = a_2 \equiv a$ and $z_1 = z_2 \equiv z$; case I: ε_1, $\varepsilon_2 \neq 0$, $a_1 = a_2 \equiv a$ and $z_1 = z_2 \equiv z$; case III: $\varepsilon_1 = \varepsilon_2 = 0$, $a_1 = a_2 \equiv a$ and $z_1 \neq z_2$. We have used $x_0 = 0.5$; an initial transient of about 1000(30000) has been left out of consideration for the x vs. a graphs (λ vs. a graphs); the calculation of $x(\lambda)$ has been done with 400 (100000) points for each value of a, which was in turn increased by steps of 0.0075 (0.0025).

Fig. 3 - Evolution of the successive ratios $\{\delta_k\}$ for the $z_1 = z_2 = 2$ prototype and for the case III ($z_1 \neq z_2$).

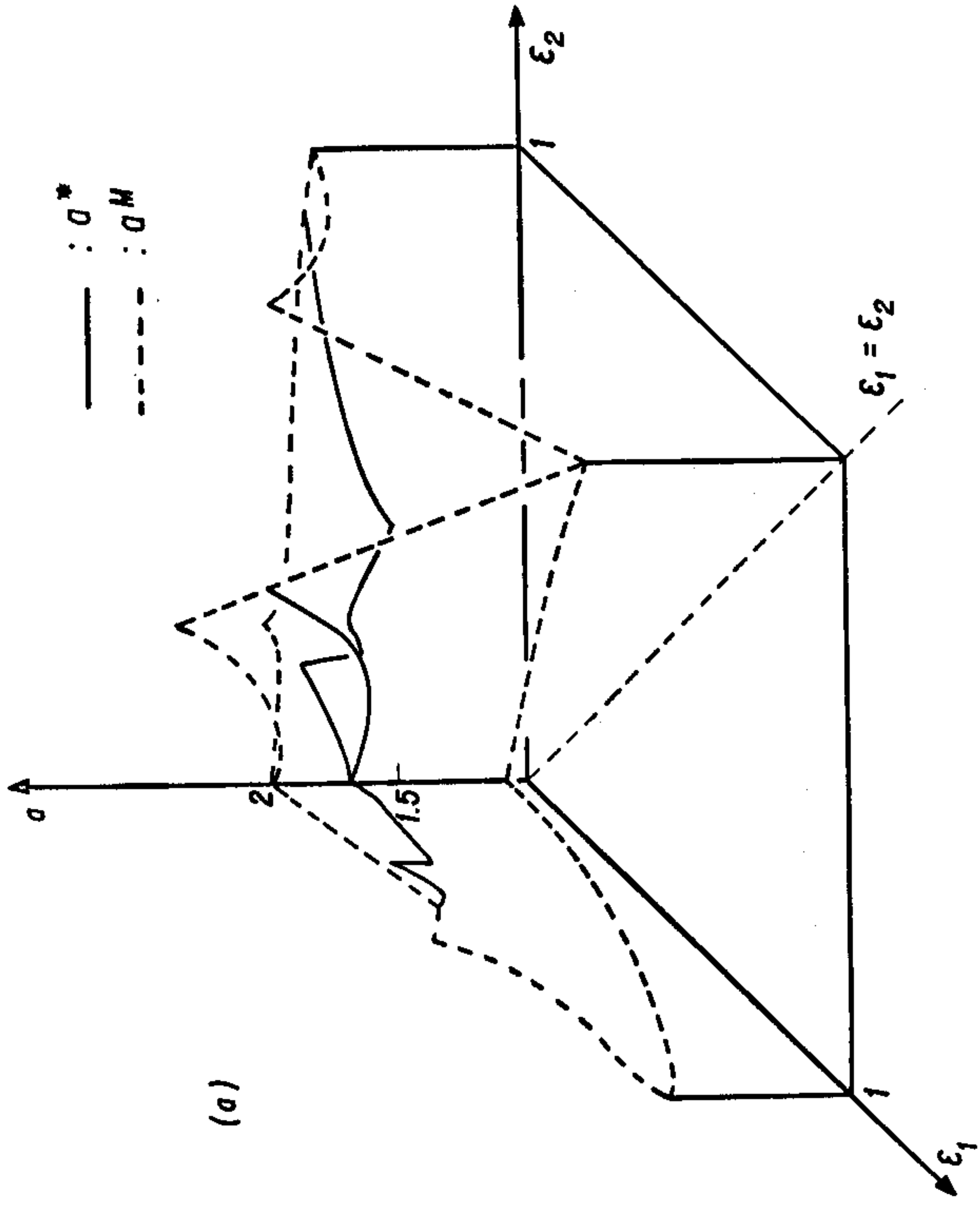

: a^*
: a^M
a
2
1.5
ε_2
1
$\varepsilon_1 = \varepsilon_2$
1
ε_1
(a)

FIG.1

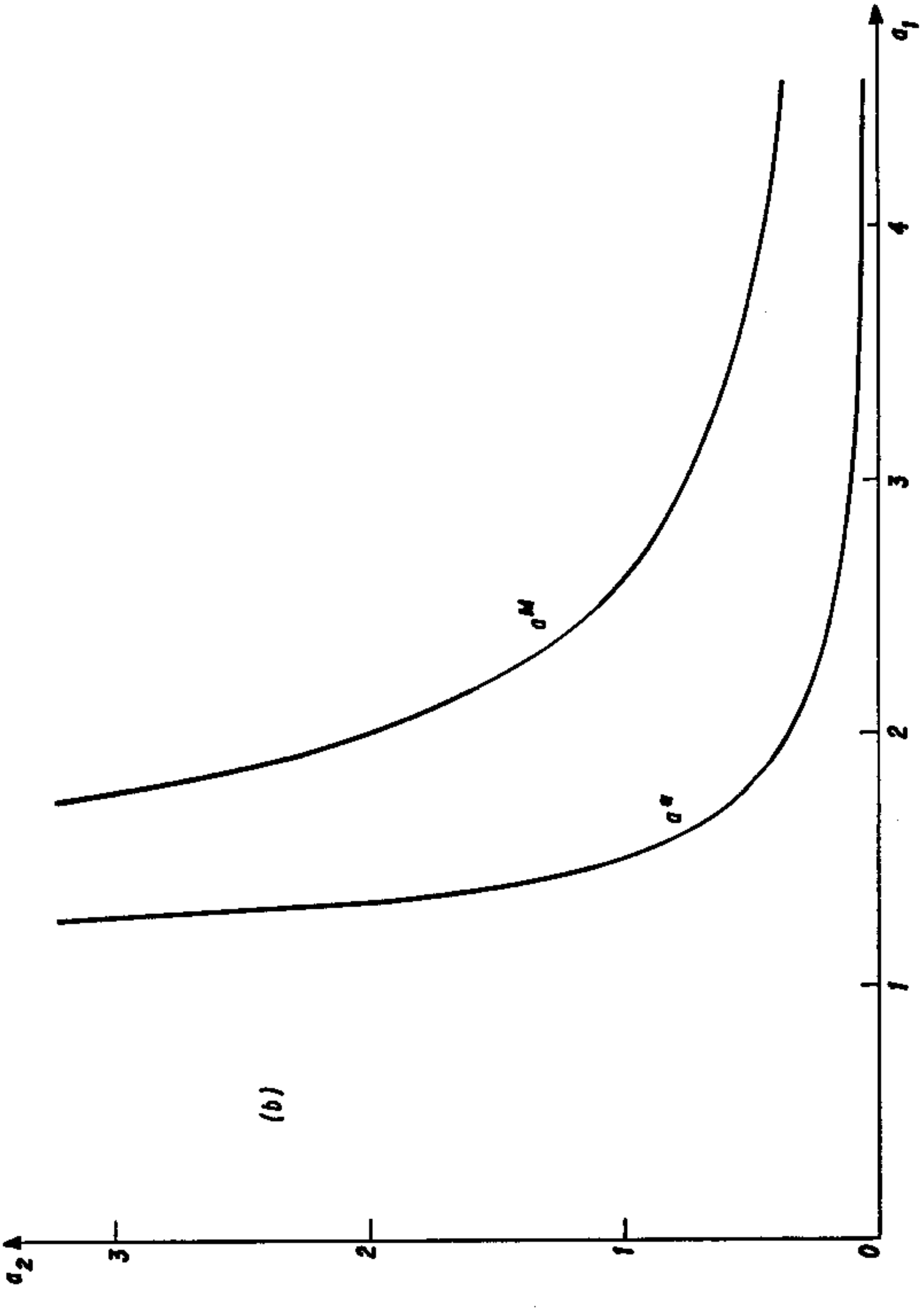
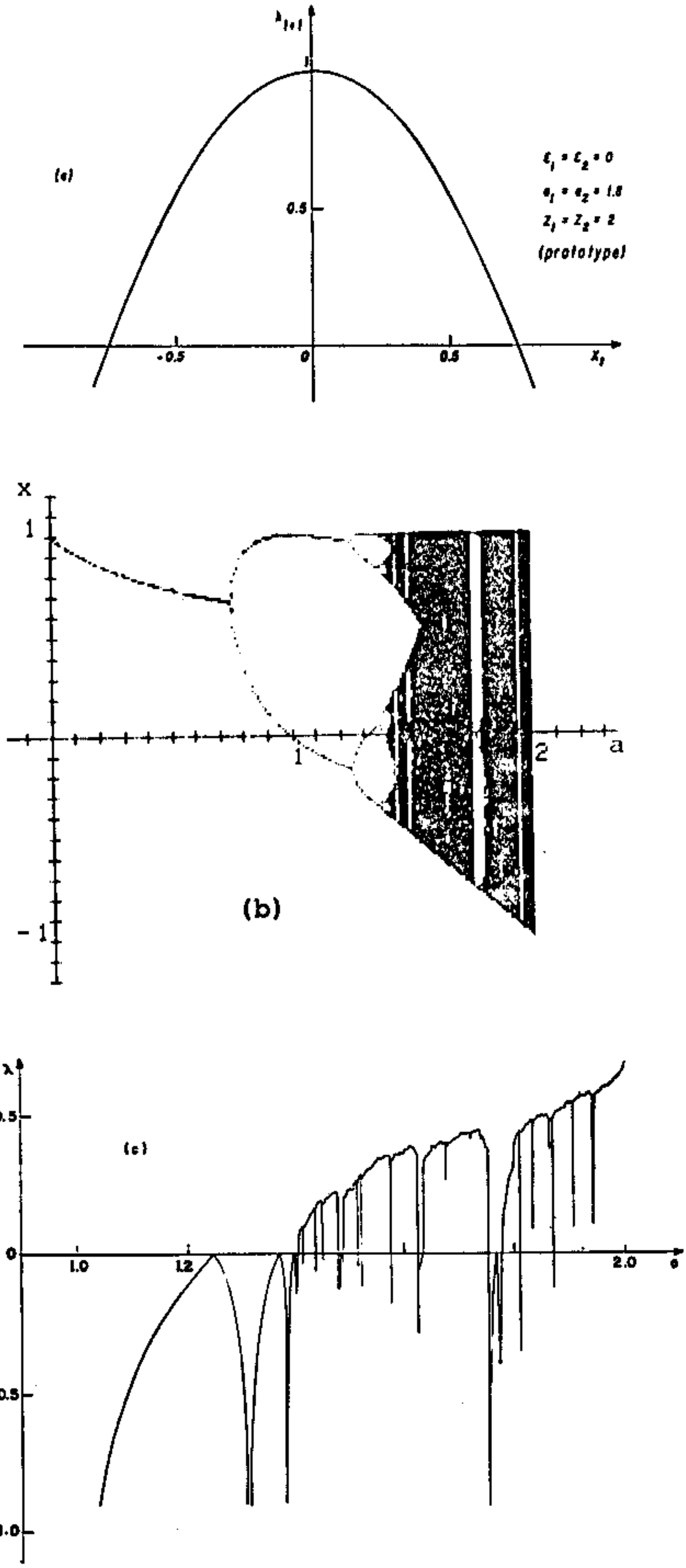
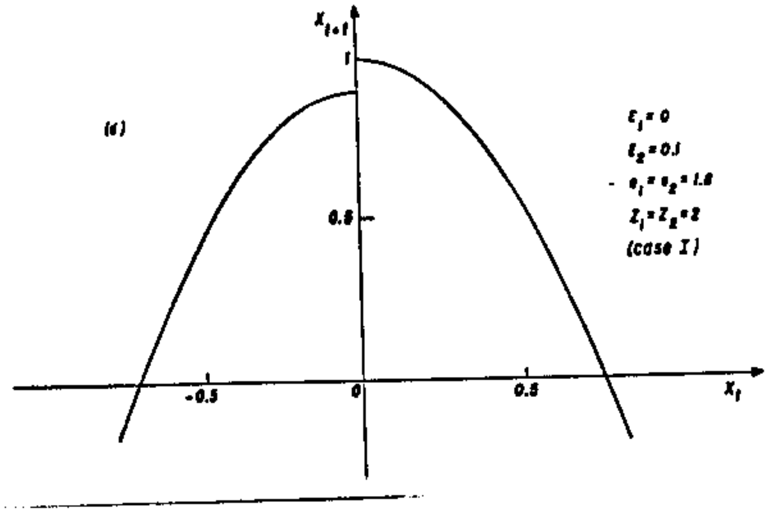
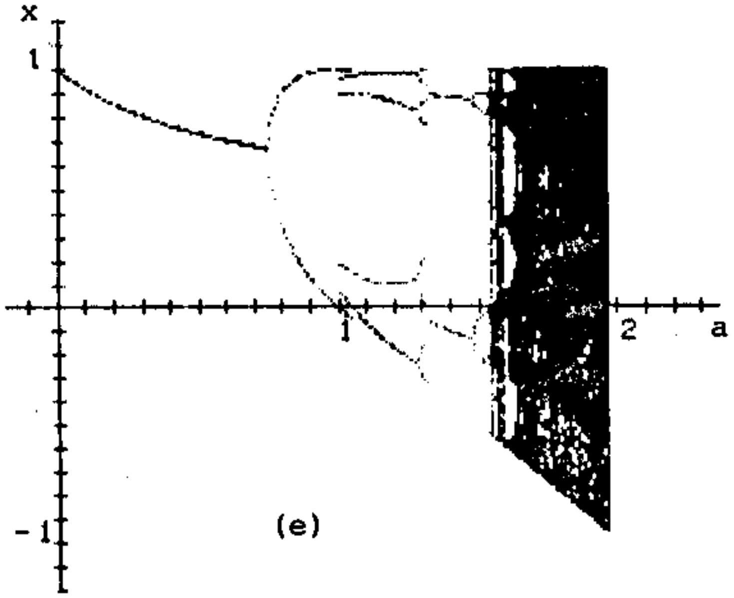
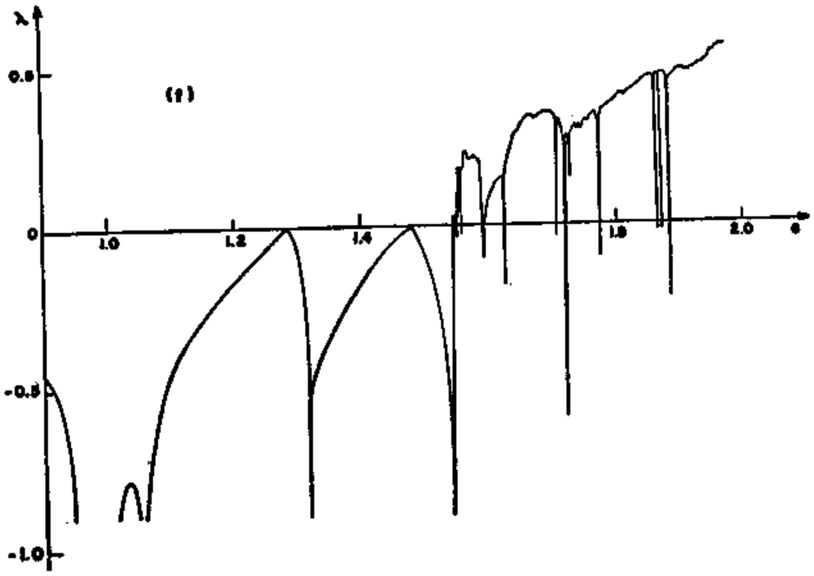


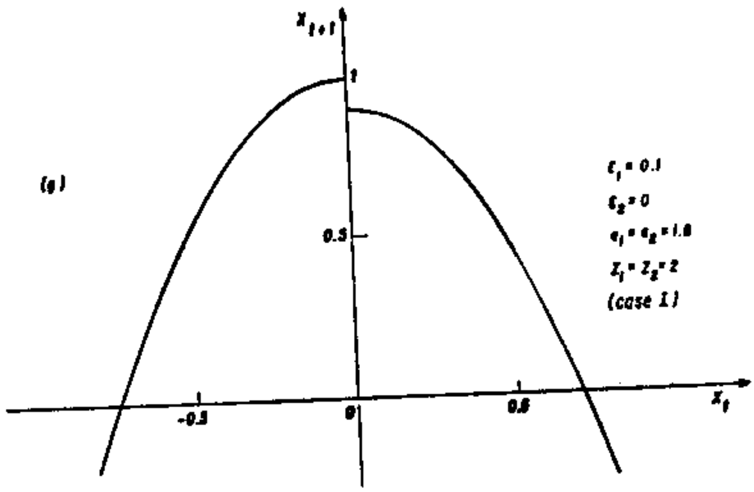
FIG.1

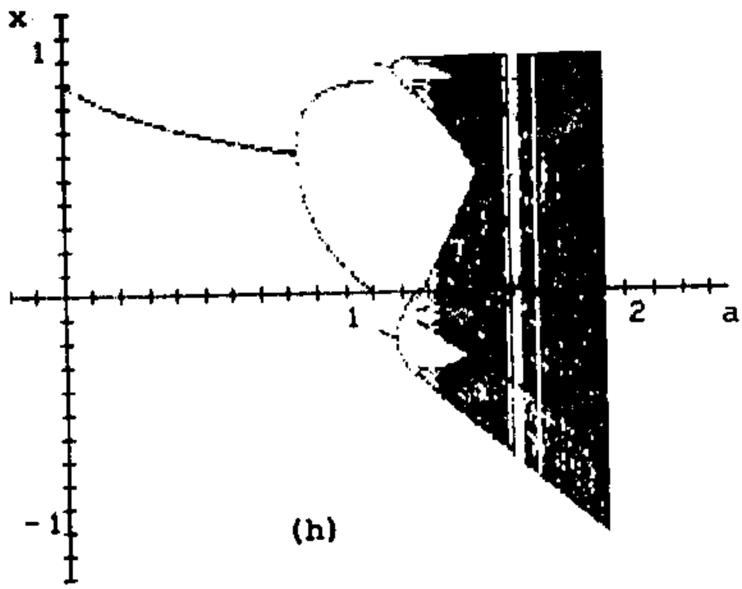

(a)
$\varepsilon_1 = \varepsilon_2 = 0$
$e_1 = e_2 = 1.8$
$z_1 = z_2 = 2$
(prototype)
0.5
-0.5
0
0.5
x_1
x
1
1
2
a
(b)
- 1
(c)
0.5
0
1.0
1.2
2.0
-0.5
-1.0

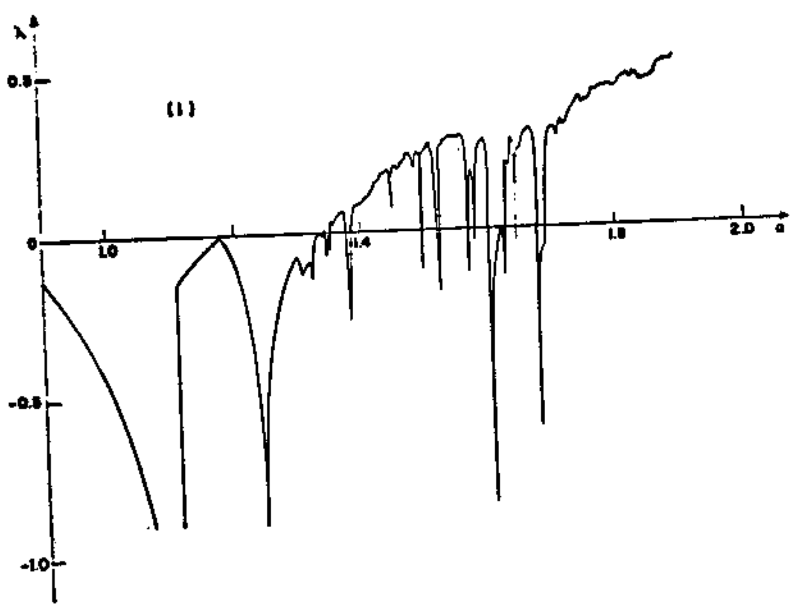

(d)
X_{t+1}
1
0.5
-0.5
0
0.5
X_t
$\varepsilon_1 = 0$
$\varepsilon_2 = 0.1$
$e_1 = e_2 = 1.8$
$Z_1 = Z_2 = 2$
(case I)

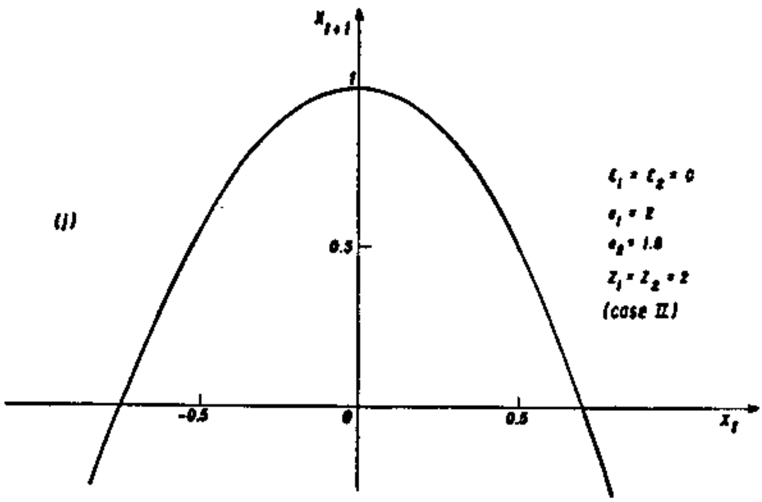

x
1
- 1
1
2
a
(e)

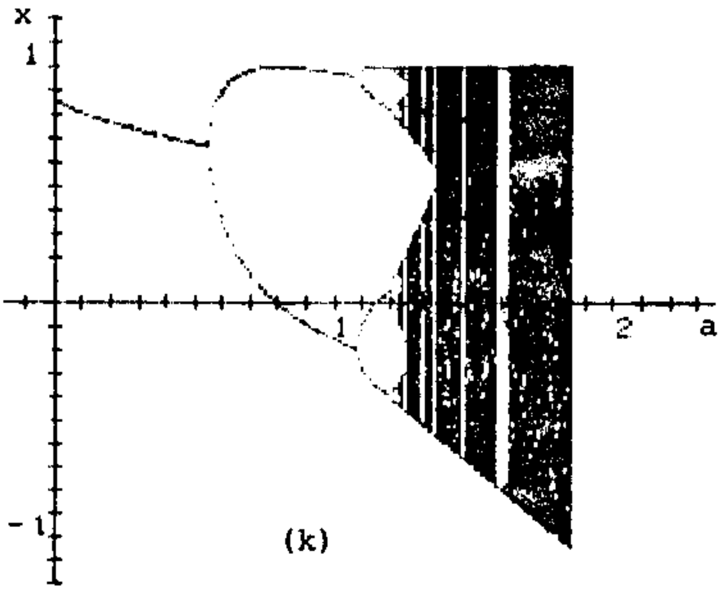

λ
0.5
(f)
0
1.0
1.2
1.4
1.8
2.0
-0.5
-1.0

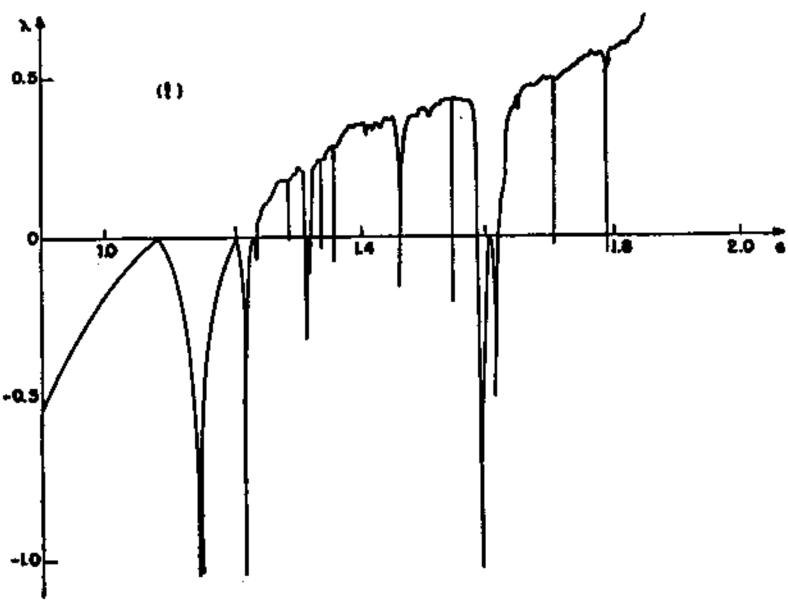

(g)
(case I)

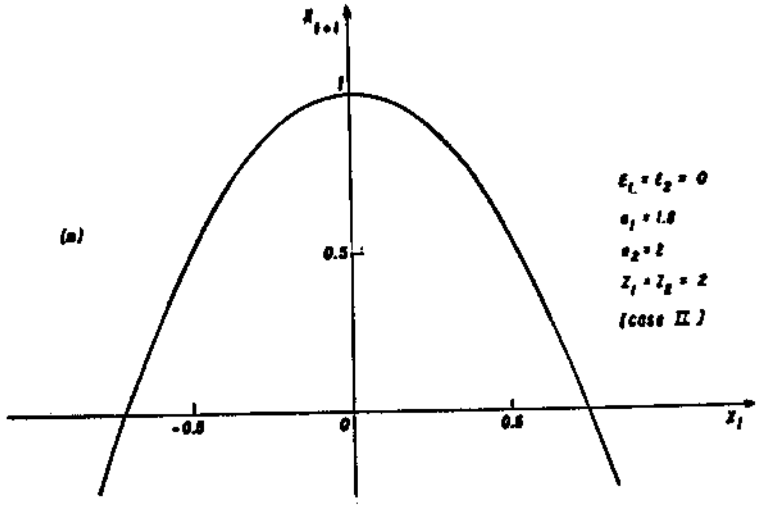

x
1
-1
1
2
a
(h)

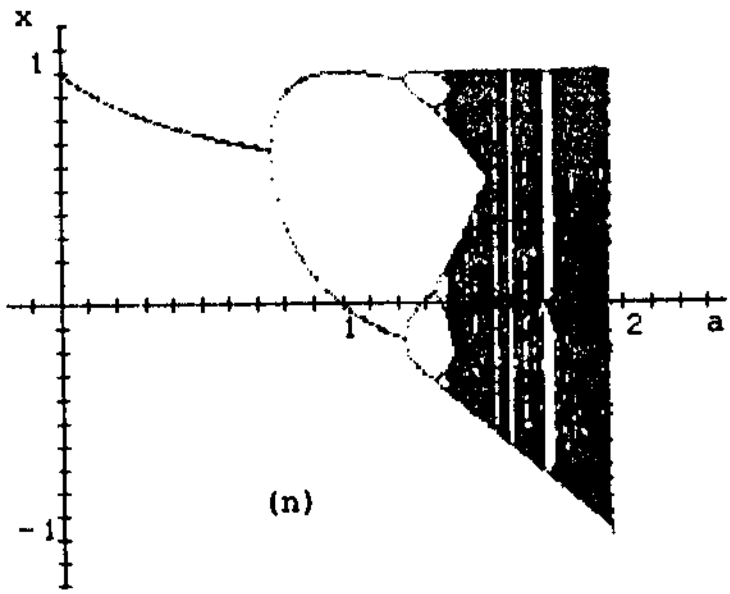

(i)
0.5
0
-0.5
-1.0
1.0
1.4
1.8
2.0

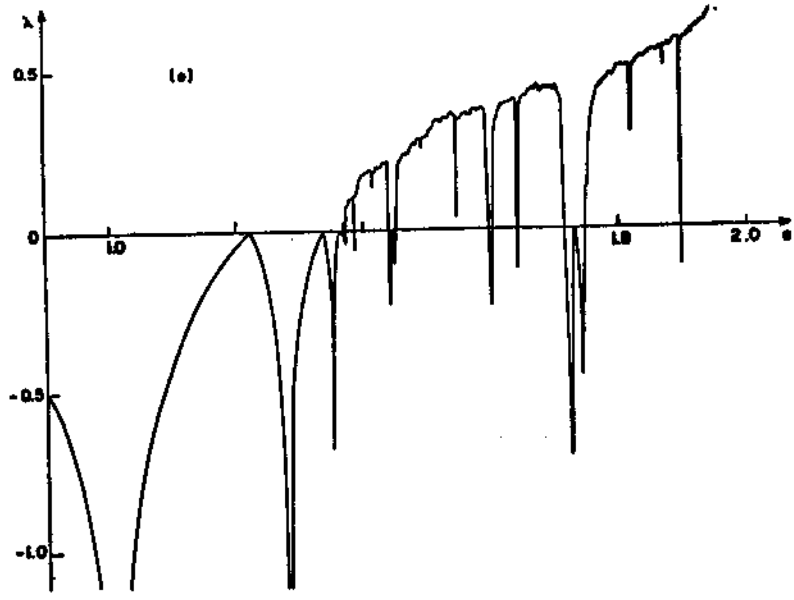

(j)
$\varepsilon_1 = \varepsilon_2 = 0$
$a_1 = 2$
$a_2 = 1.8$
$z_1 = z_2 = 2$
(case II)

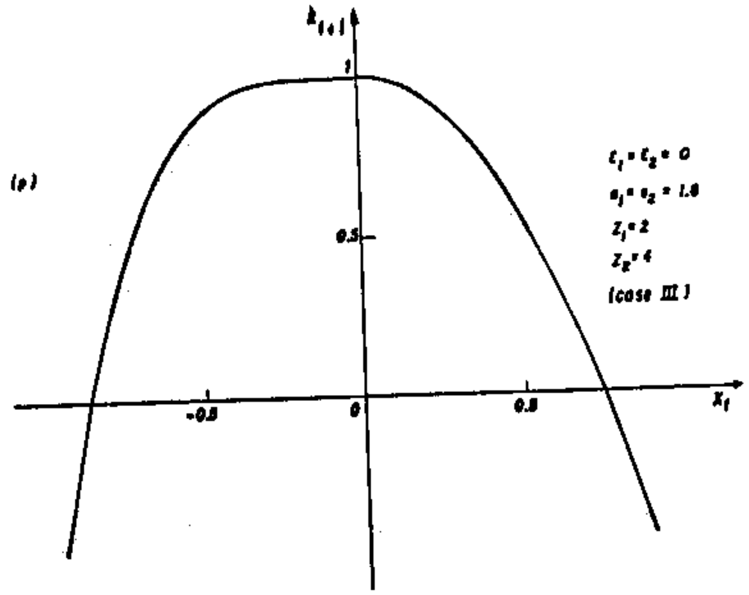

(k)

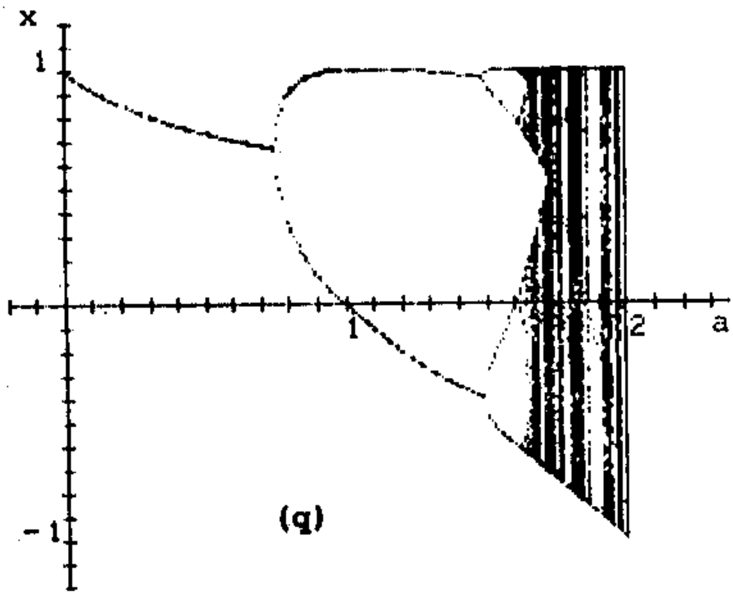

(l)

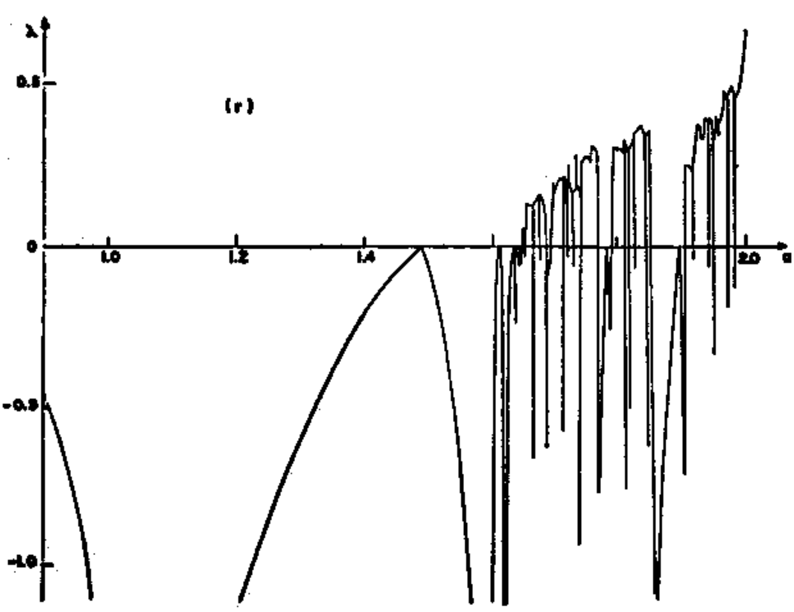



x
1
-1
1
2
a
(n)




(p)
0.5
-0.5
0
0.5
1
(case III)


x
1
-1
1
2
a
(q)


λ
0.5
0
-0.5
-1.0
1.0
1.2
1.4
2.0
a
(r)

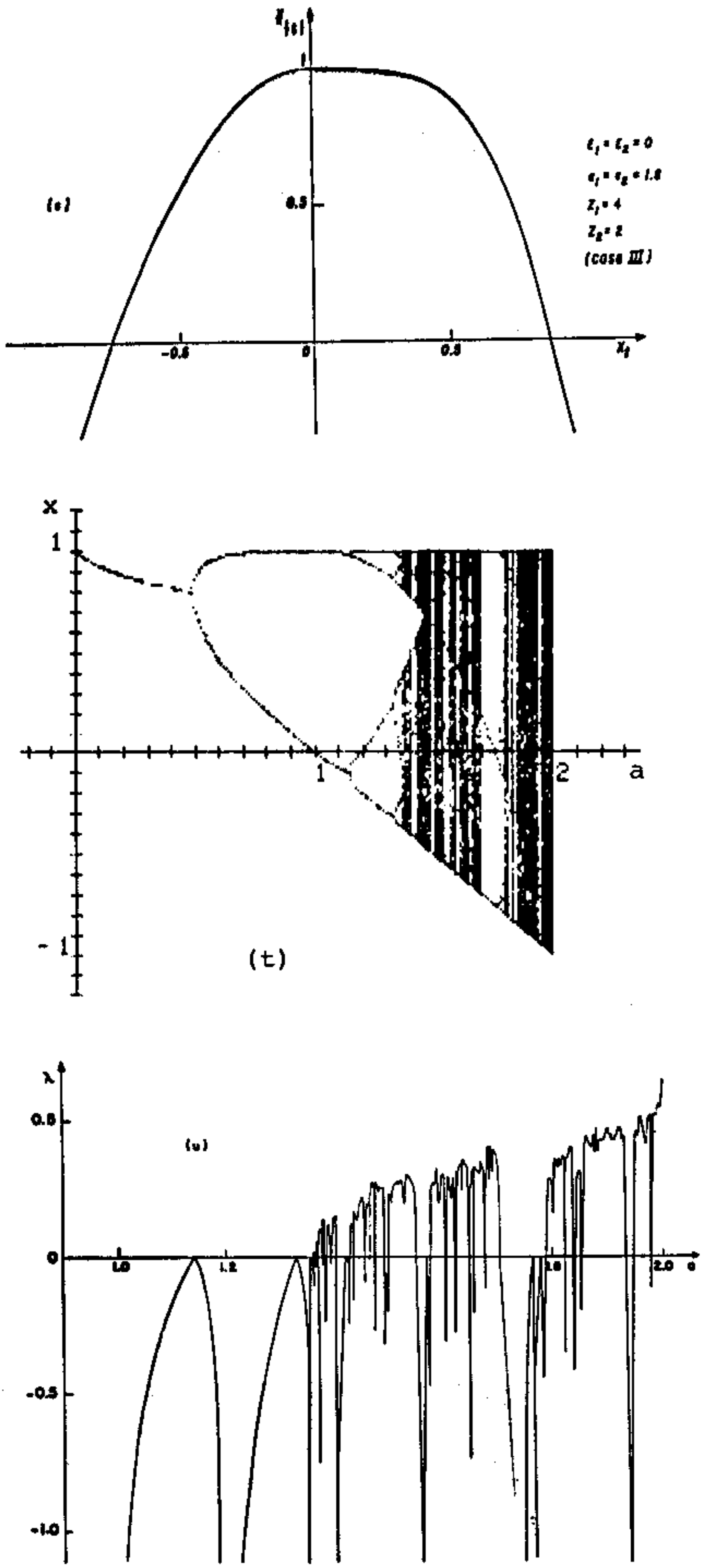


FIG. 2

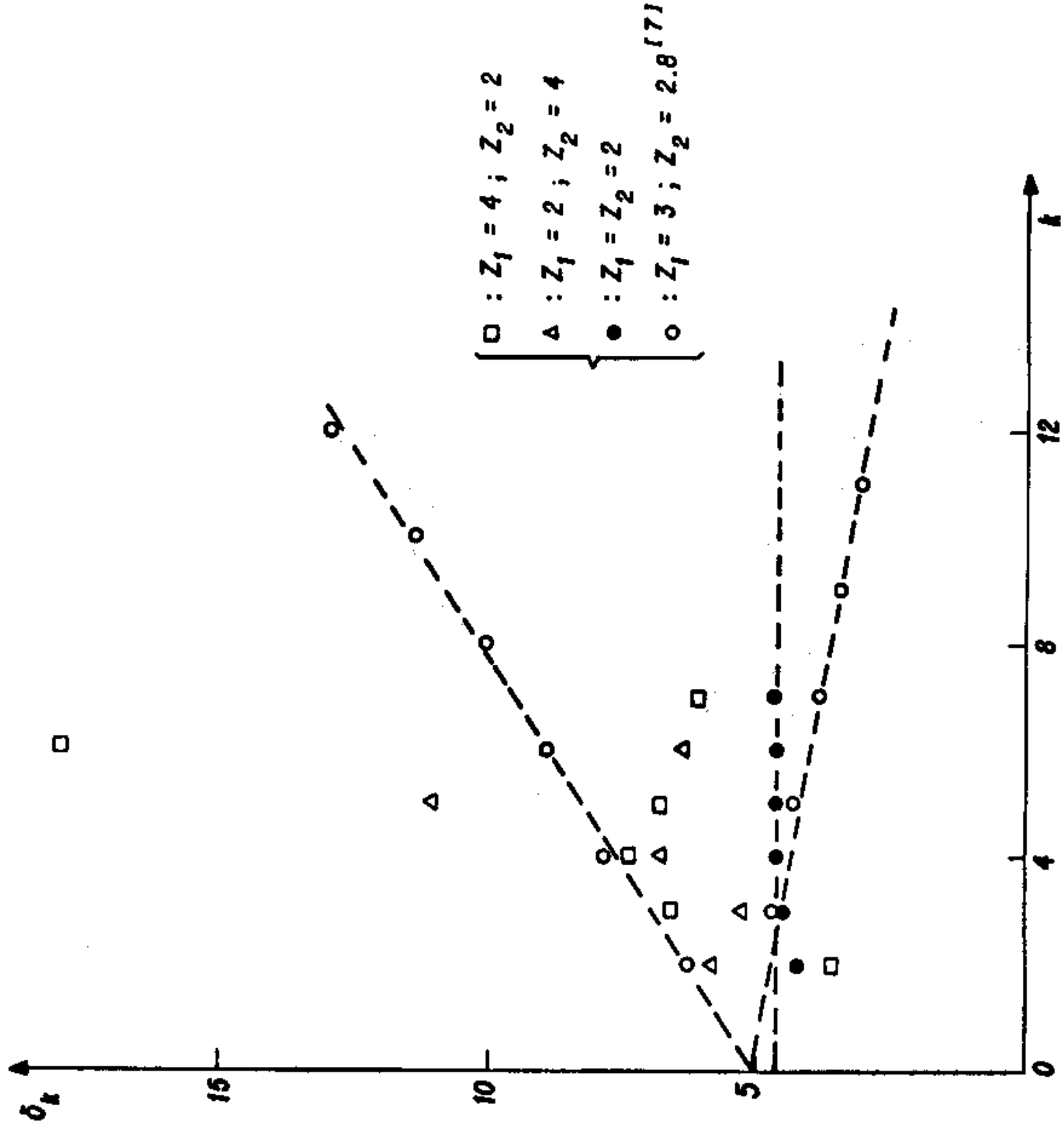


FIG. 3

REFERENCES

[1] R.M. May, Nature 261, 459 (1976).

[2] J. Maurer and A. Libchaber, J. Phys. (Paris) Lett. 40, L419 (1979); A. Libchaber and J. Maurer, J. Phys. (Paris) 41, C3-51 (1980); J.P. Gollub, S.V. Benson and J. Steinman, Ann. N.Y. Acad. Sci. 357, 22 (1981); J. Giglio, S. Musazzi and U. Perinni, Phys. Rev. Lett. 47, 243 (1981).

[3] P.R. Hauser, C. Tsallis and E.M.F. Curado, Phys. Rev. A30, 2074 (1984); Bambi Hu, Indubala I. Satija, Phys. Lett. 98A, 143 (1983).

[4] M.J. Feigenbaum, J. Stat. Phys. 19, 25 (1978); P. Coullet and C. Tresser, J. Phys. (Paris) C 5, 25 (1978).

[5] A.A. Hnilo, Optics Comm. 53, 194 (1985).

[6] G. Contopoulos, N. Cimento/Lettere 37, 149 (1983).

[7] R.V. Jensen and L.K.H. Ma, Phys. Rev. A 31, 3993 (1985).