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ON THE PROPAGATION OF EINSTEIN'S EQUATIONS WITH QUASI-MAXWELLIAN EQUATIONS OF GRAVITY

by

M. Novello and J.M. Salim

Centro Brasileiro de Pesquisas Físicas - CNPq/CBPF
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

In a recent letter to this journal B. Lesche and M. Som[1] (henceforth called PL1) claimed that they have found "the extra condition necessary to propagate Einstein's equations from a space-like hypersurface into space-time with quasi-Maxwellian equations". Besides this they pretend that since the pioneering work on this field until to-day, people have missed that there is no equivalence in the use of Weyl conformal tensor instead of the full curvature tensor in Bianchi identities regarded as the equation of evolution — contrary to a largely spread belief[2,3]. These are the two pieces which constitutes Lesche and Som's paper. We would like to point out here that:

(i) the first affirmation which is contained in the form of a Theorem, is correct — although it is well-known — and has been ennunciated and demonstrated by A. Lichnerowicz some twenty four years ago[4]; and that

(ii) the second claim is simply wrong.

The question is the following: if we take Bianchi identities

(1a) $$R^{\alpha\beta\mu\nu}{}_{;\nu} = R^{\mu[\alpha;\beta]}$$

and add to it the evolution equation

(1b) $$R^{\alpha\beta\mu\nu}{}_{;\nu} = -k\, T^{\mu[\alpha;\beta]} + \frac{k}{2}\, g^{\mu[\alpha}T^{,\beta]}$$

under what conditions will this set (1a,b) of equations be completely equivalent to Einstein's equations ? In PL-1 it is claimed that Lichnerowicz gave the answer to this question in the case of vacuum by arguing that eq. (1a,b) must be implemented by the conditions that Einstein's equation in the vacuum must be valid in a space like hypersurface Σ, $R_{\mu\nu}(\Sigma) = 0$. Lesche and Som argue that in presence of source there is a missing piece of information to assurre the propagation equation. This is supplied by the condition they propose

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{2}$$

Then they state a theorem valid for the source case.

We would like to point out that the statement and the demonstration of this theorem, including condition (2) is precisely contained in a theorem by A. Lichnerowicz which appeared in 1961[4].

Now let us turn to the second point of PL1.

Let us deal with Weyl conformal tensor $C^{\alpha\beta\mu\nu}$. We have,

$$C^{\alpha\beta\mu\nu}{}_{;\nu} = \frac{1}{2} R^{\mu[\alpha;\beta]} - \frac{1}{12} g^{\mu[\alpha}R^{,\beta]} \tag{3a}$$

$$C^{\alpha\beta\mu\nu}{}_{;\nu} = - \frac{k}{2} T^{\mu[\alpha;\beta]} + \frac{k}{6} g^{\mu[\alpha}T^{,\beta]} \tag{3b}$$

From this set of equations we obtain

$$(R_{\mu\alpha} - \frac{1}{6} Rg_{\mu\alpha} + kT_{\mu\alpha} - \frac{k}{3} Tg_{\mu\alpha})_{;\beta} - \tag{4}$$

$$- (R_{\mu\beta} - \frac{1}{6} Rg_{\mu\beta} + kT_{\mu\beta} - \frac{k}{3} Tg_{\mu\beta})_{;\alpha} = 0$$

First of all let us remind the reader that it is a trivial exercise to show that

(5) $$R_{\mu\alpha} - \frac{1}{6} Rg_{\mu\alpha} + kT_{\mu\alpha} - \frac{k}{3} Tg_{\mu\alpha} = 0$$

is completely equivalent to Einstein's equation, that is

(5)' $$R_{\mu\alpha} - \frac{1}{2} Rg_{\mu\alpha} + kT_{\mu\alpha} = 0$$

Thus, let us impose that (5) or equivalently (5)' vanishes on Σ.

Then from (4) we have that throughout the whole space-time we have:

(6) $$R_{\mu\ell} - \frac{1}{2} Rg_{\mu\ell} + kT_{\mu\ell} = 0$$

for $\mu = 0,1,2,3$ and $\ell = 1,2,3$. In order to arrive at (6) use (5) into (4) and make $\beta = 0$ and $\alpha = \ell$ for arbitrary μ .

Now using the conservation of $T_{\mu\nu}$ and expression (6) we obtain that

(7) $$kT^{\mu 0}{}_{;0} - (R^{\mu\ell} - \frac{1}{2} Rg^{\mu\ell})_{;\ell} = 0$$

Now, contracted Bianchi identity allows us to write (7) in the form

$$kT^{\mu 0}{}_{;0} + (R^{\mu 0} - \frac{1}{2} Rg^{\mu 0})_{;0} = 0$$

This expression is valid throughout the whole space-time. Applying it to Σ we obtain that

$$R^{\mu 0} - \frac{1}{2} Rg^{\mu 0} + kT^{\mu 0} = 0$$

which ends the proof . That is, contrary to the claim of PL-1, equations (3a,b) are completely equivalent to (1a,b) under the condition of conservation of energy — no further arbitrary condition is needed. Thus PL-1 in this respect, is wrong.

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