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INSTANTON, MERON AND A NON-SELF DUAL SOLUTIONS OF U(n,p) MODEL

by

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ABSTRACT:

Instanton and meron type solutions are obtained for the generalized CP^{n-1} model - U(n,p) model - in two dimensions. The equation of motion is cast in a convenient symmetrical form.

I. INTRODUCTION

The O(3) σ model[1] and CP^{n-1} models[2] in two dimensional Euclidean space-time have been of much interest recently because, like non-abelian gauge theory in four dimensions, they contain instantons and non-trivial topological structures - Generalizations of these models have been considered recently. One starts with a manifold of the kind

$$G(n,p) = \frac{G(n)}{G(p) \times G(n-p)} \tag{1}$$

with G(p) = O(p), U(p), Sp(p) etc., called Grassmann manifolds, with G(n) as the global invariance group, G(p) is the gauge group and G(n-p) is the invariance sub-group of the field. For G=U the model was discussed in Ref.3, for G=O in Ref.4, for G=Sp in Ref. 5 and HP^{n-1} model in Ref.6.

We discuss, for definiteness sake, the classical solutions for the case of U(n,p) model. Instanton, meron and a non-self dual solution are obtained.

II. U(n,p) MODEL:

The action for U(n,p) model may be written in terms of complex scalar fields Z_a^{α},($a = 1...n; \alpha = 1...p$) taking values in the Grassmann manifold U(n,p). The fields are subject to constraints

$$Z^{+}Z = I_p \tag{2}$$

where $Z = (Z_a^{\alpha})$ is a (n x p) matrix and I_p is (p x p) identity matrix. The fields transform as

$$Z \longrightarrow VZU \tag{3}$$

where $V \varepsilon U(n)$, the global invariance group and $U(x) \varepsilon U(p)$, the gauge group. The Lagrangean density is

$$\begin{aligned} L &= Tr \left[(D_\mu Z)^+(D_\mu Z)\right] \\ &= Tr \left[(\partial_\mu Z^+)\ (\partial_\mu Z) + (Z^+\partial_\mu Z)^2\right] \end{aligned} \tag{4}$$

where we introduce a matrix gauge potential $(A_\mu)_{\alpha\beta}$ defined by

$$D_\mu Z \equiv (\partial_\mu Z + iZA_\mu) \tag{5}$$

then

$$A_\mu = A_\mu{}^+ = iZ^+\partial_\mu Z \tag{6}$$

L is invariant under local gauge transformations

$$Z \rightarrow ZU \quad , \quad A_\mu \rightarrow U^+A_\mu U + iU^+\partial_\mu U \tag{7}$$

and under global transformations

$$Z \rightarrow VZ \quad , \quad A_\mu \rightarrow A_\mu \tag{8}$$

For $p \neq 1$ we have non-abelian gauge group. Under gauge transformations the gauge covariant derivatives transform as $D_\mu Z \rightarrow (D_\mu Z)U$

and $D_\mu D_\nu Z = \partial_\mu (D_\nu Z) + i(D_\nu Z)A_\mu \rightarrow (D_\mu D_\nu Z)U$.

Introducing Lagrange multiplies fields to take case of the constraints we derive the equations of motion to be

$$D_\mu D_\mu Z + Z(D_\mu Z)^+(D_\mu Z) = 0 \tag{9}$$

The conserved Noether current corresponding to the global invariance group is found to be $j_\mu = [M, \partial_\mu M]$ where $M = ZZ^+$ is gauge invariant with $TrM = p$. An infinite set of non - local conserved currents (in two dimensions) may then be constructed(7).

The O(n) non-linear σ model has the drawback that there is only a non-trivial topological structure for $n = 3$, the instantons and anti-instantons, fulfilling the self-duality equations. In contrast CP^{n-1} model(2)(3) and its generalization U(n,p) contain always a non-trivial topological structure. The topological charge $\tilde{Q}$ of a solution may be defined as

$$\tilde{Q} = \frac{1}{2\pi} \int Q(x)\, d^2x \tag{10}$$

where the topological charge density is:

$$\begin{aligned} Q(x) &= i\, \varepsilon_{\mu\nu} \left[Tr\ (D_\mu Z)^+(D_\nu Z) \right] \\ &= \partial_\mu \left[\varepsilon_{\mu\nu}\ Tr\ A_\nu \right] \end{aligned} \tag{11}$$

The self duality equations in U(n,p) model are †

$$D_{\mp} Z = 0 \tag{12}$$

† $x_\pm = (x_1 \pm i x_2)$, $\partial_\pm = \frac{1}{2} [\partial_1 \mp i \partial_2]$, $D_\pm = \frac{1}{2} [I - ZZ^+]\ \partial_\pm Z$

and give rise to finite action instanton and anti-instanton solutions. We will also find below non-self dual meron solutions as well as another non-self dual solution in parallel to those found for CP^{n-1} in Ref.8. The relevance of non-self dual solution for quantized theory has been emphasized in recent publications.

We show easily

$$L = 2\,\mathrm{Tr}\left[|D_-Z|^2 + |D_+Z|^2\right]$$

$$Q(x) = 2\,\mathrm{Tr}\left[|D_-Z|^2 - |D_+Z|^2\right] \tag{13}$$

and the equations of motion take the form

$$D_+D_-Z + Z\,|D_-Z|^2 = 0$$

or

$$D_-D_+Z + Z\,|D_+Z|^2 = 0 \tag{14}$$

if we use the identity

$$(D_-D_+ - D_+D_-)\,Z + Z\left[|D_+Z|^2 - |D_-Z|^2\right] = 0 \tag{15}$$

The energy momentun tensor is

$$\tau_{11} = -\tau_{22} = 2\mathrm{Tr}\left[(D_+Z)^+(D_-Z) + (D_-Z)^+(D_+Z)\right]$$

$$\tau_{12} = \tau_{21} = 2i\mathrm{Tr}\left[(D_-Z)^+(D_+Z) - (D_+Z)^+(D-Z)\right] \tag{16}$$

and energy momentum conservation leads to

$$\partial_+ \, \mathrm{Tr} \left[(D_+Z)^+ (D_-Z) \right] = 0$$

$$\partial_- \, \mathrm{Tr} \left[(D_-Z)^+ (D_+Z) \right] = 0 \qquad (17)$$

To obtain non-self dual solutions it is convenient to work with uncontrained field $\hat{Z}$ defined by

$$Z = \hat{Z} \frac{1}{|\hat{Z}|} \qquad (18)$$

where $|\hat{Z}|^2 = \hat{Z}^+\hat{Z}$ is a (pxp) matrix.
We write $P = \hat{Z} \frac{1}{|\hat{Z}|^2} \hat{Z}^+$ which satisfies $P^2=P=P^+$ and $(I_n-P)\hat{Z}= 0$
The equations of motion give

$$(I_n-P) \left[\partial_+\partial_-\hat{Z} - (\partial_+\hat{Z}) \frac{1}{|\hat{Z}|^2} \hat{Z}^+(\partial_-\hat{Z}) - (\partial_-\hat{Z}) \frac{1}{|\hat{Z}|^2} \hat{Z}^+\partial_+\hat{Z} \right] = 0 \qquad (19)$$

and A_μ takes the form

$$A_\mu = i \frac{1}{|\hat{Z}|} \left[\hat{Z}^+(\partial_\mu \hat{Z}) - |\hat{Z}| \, \partial_\mu |\hat{Z}| \right] \frac{1}{|\hat{Z}|} \qquad (20)$$

while

$$D_\pm Z = (I-P)(\partial_\pm \hat{Z}) \frac{1}{|\hat{Z}|} \qquad (21)$$

We remark that it is possible to make use of the gauge invariance of the theory to parametrize(3) the coset space in terms of p(n-p) complex fields K and write

$$\hat{Z} = \begin{pmatrix} K \\ I_p \end{pmatrix} \tag{22}$$

Making use of the fact that (ZZ^+) transforms linearly under U(n) and in gauge invariant we may readily derive the non-linear transformation properties of K which transform linearly under U(p)xU(n-p) subgroup. The Lagrangian takes the form

$$L = \frac{1}{2} \mathrm{Tr} \left[L^2 (\partial_\mu K^+) \; H^2 \; (\partial_\mu K) \right] \tag{23}$$

where $L^2(I_p + K^+K) = I_p$, $H^2 = (I_{(n-p)} - KL^2K^+)$ and we may define covariant derivative of K in the sense of non - linear realizations(9) as

$$\mathbb{D}^\mu K = H(\partial_\mu K) \tag{24}$$

This parametrization, however, is not convenient for obtaining non-self dual solutions.

III. INSTANTONS, MERONS and a CLASS of NON-SELF DUAL SOLUTIONS

Instanton solutions for CP^{n-1} model (p=1) have been widely discussed in literature.

In the case of U(n,p) a 1-instanton solution may be written as $Z_a^{\alpha} = (x_+ - b_a^{\alpha})$.

We find $A_\mu \xrightarrow{|x|\to\infty} iU^+\partial_\mu U$ where $U = e^{2\theta I_p}$, $\theta = \arg (x_1 + ix_2)$ and $\tilde{Q} = p$.

A meron solution for CP^{n-1} model is written as

$$Z = \frac{1}{\sqrt{2}} \left[f(x)u + v \right] \tag{25}$$

where $|f|^2 = 1$ and u,v are constant vector satisfying $u^+u = v^+v = 1$, $u^+v = 0$. We find

$$\partial_-\partial_+ f + \frac{1}{2} \left[|\partial_+ f|^2 + |\partial_- f|^2 \right] f = 0 \tag{26}$$

It is clear that only non-self dual solutions are obtained in this form. Choosing, for example,

$$f = \sqrt{\frac{(x_+ - \alpha)(x_- - \beta^*)}{(x_- - \alpha^*)(x_+ - \beta)}} \tag{27}$$

we find

$$L = \frac{1}{4} \left| \frac{1}{(x_+ - \alpha)} - \frac{1}{(x_+ - \beta)} \right|^2 \tag{28}$$

$$Q = -\frac{1}{2} \bar{\nabla} \cdot \left[\frac{(\bar{x} - \bar{\alpha})}{(\bar{x} - \bar{\alpha})^2} - \frac{(\bar{x} - \bar{\beta})}{(\bar{x} - \bar{\beta})^2} \right] \tag{29}$$

$$= - \Pi \left[\delta^2(\bar{x} - \bar{\alpha}) - \delta^2(\bar{x} - \bar{\beta}) \right]$$

where $\bar{x} = (x_1, x_2)$ $\bar{\nabla} = (\partial_1, \partial_2)$. For n-meron configuration we may take

$$f = \prod_{i=1}^{n} \sqrt{\frac{(x_+ - \alpha_i)}{(x_- - \alpha_i^*)}} \tag{30}$$

For the U(n,p) model we will illustrate the procedure for n=3, p=2. Write the 3 x 2 matrix Z as $Z = (Z_1, Z_2)$ where

$Z_{1,2}$ are 3-component column vectors. The constraints are then given as $Z_1^\dagger Z_1 = Z_2^\dagger Z_2 = 1$, $Z_1^\dagger Z_2 = 0$, and $P = Z_1 Z_1^\dagger + Z_2 Z_2^\dagger$; $(I - P)\partial_\pm Z_1 = (I - P)(\partial_\pm \hat{Z}_1) |\hat{Z}_1|^{-1}$

The equations of motion are easily written in terms of $Z_{1,2}$. Writing $\hat{Z}_1 = (f\, u + \lambda\, v)$, $Z_2 = g\, w$ where u,v,w constitute an orthonormal set of constant vectors we find

$$\partial_-\partial_+ f - \frac{1}{(\lambda^2+|f|^2)} \left[(\partial_+ f) f^*(\partial_- f) + (\partial_- f) f^*(\partial_+ f) \right] = 0 \tag{31}$$

For the self-dual solution corresponding to $D_- Z = 0$ we obtain $\partial_- f = 0$. Choosing $f = (x_+ - \alpha)^m$, for example, a finite action is obtained for $m \geq 1$ and we get $S = 2\pi\tilde{Q} = -2\pi m$. A meron solution is obtained for $|f|^2 = \lambda^2$ so that the Eq. (31) reduces to Eq. (26). The action is infinite and topological density is concentrated at isolated points.

Finally we remark that a non-self-dual solution may be obtained starting from a nxp matrix satisfying, say, $\partial_- F = 0$ and $F^\dagger F \neq$ const. We verify that

$$\hat{Z} = (I - F \frac{1}{|F|^2} F^+)\partial_+ F \tag{32}$$

satisfies Eq. 19 and $D_- Z = -F|F|^{-2}|\bar{Z}|$, $D_+ Z = (I - ZZ^\dagger)(\partial_+^2 F)|\bar{Z}|^{-1}$ and $|\hat{Z}|^2 = \partial_-\partial_+|F|^2 - (\partial_-)|F|^2)\ |F|^{-2}\ (\partial_+|F|^2)$.

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