ISSN 0029-3865

CBPF-NF-013/86

PHENOMENOLOGICAL QUANTIZATION SCHEME IN NONLINEAR SCHRÖDINGER EQUATION

by

J.N. Maki and T. Kodama

Centro Brasileiro de Pesquisas Físicas - CNPq/CBPF
Rua Dr. Xavier Sigaud, 150 - Urca
22290 - Rio de Janeiro, RJ - Brasil

ABSTRACT

A phenomenological quantization scheme is carried out for the collision process of two identical solitons of the nonlinear Schrödinger equation. The effective potential is determined from the classical expression for the time-delay, and the corresponding S matrix is calculated for the quantum scattering of the two centers of masses of solitons. It is shown that this S matriz reproduces the exact S matrix of the quantum many body problem as well as the ground state binding energy of the system for large particle number, but it is not adequate to describe the scattering process for low energy region.

PACS nº 03.65.Nk

Key-Words: Nonlinear Schrödinger equation; Quantization.

It is well known that there are some classes of nonlinear wave equations which are exactly soluble[1]. Although all of them are one-dimensional, they offer a nice site to study the theoretical foundation of quantization of nonlinear fields. In particular, the nonlinear Schrödinger equation (NSE),

$$i\hbar \frac{\partial}{\partial t} \psi + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi + \varepsilon^2 |\psi|^2 \psi = 0 \tag{1}$$

has been studied by many authors in connection with the non-relativistic quantum mechanical many body problem[2]. The quantized version of Eq. (1) is equivalent to a system of bosons whose Hamiltonian is

$$H = \sum_{i=1}^{N} \left(-\frac{\hbar^2}{2m}\right) \frac{\partial^2}{\partial x_i^2} + \frac{2m}{\hbar^2} \varepsilon^2 \sum_{i>j} \delta(x_i - x_j) \tag{2}$$

where N is the total number of particles in the system and m is the mass of the particle.

Eq. (1) exhibits a family of soliton solutions[3]. For example, the one-soliton solution is given as

$$\psi_{1-s}(x,t) = \frac{2\hbar\eta}{\sqrt{\varepsilon^2 m}} \frac{\exp\{2i\xi(x-\bar{x}_0) - \frac{2i\hbar}{m}(\xi^2-\eta^2)t\}}{\cosh\{2\eta(x-x_0) + \frac{2\hbar\xi}{m} t\}} \tag{3}$$

and the two soliton solution

$$\psi_{2-s}(x,t) = \frac{4\hbar\eta}{\sqrt{\varepsilon^2 m}} \exp\left[-\frac{2i\hbar}{m}(\xi^2-\eta^2)t\right] \times$$

$$\times \{\exp(-2i\xi x + i\bar{x}_{01})[\exp(-2\eta(x-x_{01} + \frac{2\hbar\xi}{m} t)) - \frac{\xi^2}{(\eta-i\xi)^2} \exp(-4\eta(x-x_{02}) - \frac{4\hbar\xi}{m} t))] +$$

$$+ \exp(+2i\xi x+i\bar{x}_{02})[\exp(-2\eta(x-x_{02} - \frac{2\hbar\xi}{m}t)) - \frac{\xi^2}{(\eta+i\xi)^2}\exp(-4\eta(x-x_{01}+\frac{4\hbar\xi}{m}t))]\}$$

$$\times \{1 + \exp(-4\eta(x-x_{01}) + \frac{8\eta\xi t}{m}) + \exp(-4\eta(x-x_{02}) - \frac{8\eta\xi}{m}t) -$$

$$- 2\eta^2 \exp(-4\eta(x-x_{01}-x_{02}))\ [\frac{\xi^2-\eta^2}{(\xi^2+\eta^2)^2}\cos(4\xi x+\bar{x}_{01}-\bar{x}_{02}) + \frac{2\xi\eta}{(\xi^2+\eta^2)^2} \times$$

$$\times \operatorname{sen}(4\xi x+\bar{x}_{01}-\bar{x}_{02})] + \frac{\xi^2}{\xi^2+\eta^2}\exp(-4\eta(2x-x_{01}-x_{02}))\}^{-1} \quad . \tag{4}$$

where η, ξ, x_0, and $\bar{x}_0$ are arbitrary constants. It is easy to see that the two-solitons solution, Eq. (4), splits into two one--soliton solutions in the limit of $|t| \to \infty$. Eq. (4) represents the scattering process of two identical solitary waves in their C.M. system with the relative kinetic energy given by[4]

$$E = \frac{16\,\hbar^4\,\eta\,\xi^2}{m^2\varepsilon^2} \tag{5}$$

In this case, the time-delay caused by the interaction between the two solitary waves is calculated as

$$\Delta t = -\frac{m}{4\hbar\eta\xi}\,\ell n(1 + \frac{\eta^2}{\xi^2})$$
$$= -\frac{4\hbar^3}{m\varepsilon^4}\,\frac{1}{aN}\,\ell n(1 + \frac{N^2}{a^2}) \tag{6}$$

In Eq. (6), for the sake of later convenience, we have introduced new variables, a and N, instead of ξ and η defined by

$$N = \frac{4\hbar^2\eta}{m\varepsilon^2} \tag{7}$$

$$a = 2\sqrt{\frac{\hbar^2 E}{mN\varepsilon^4}} \quad . \tag{8}$$

N is related to the normalization integral of the system as

$$2N = \int_{-\infty}^{\infty} |\psi_{2-s}|^2 \, dx \quad , \tag{9}$$

so that it is regarded as the number of particles contained in each solitary wavepacket in the asymptotic region.

In the quantized version, the above two-soliton scattering is considered as the collision of two bound states of N-particle systems with the translational kinetic energy E. The exact quantum mechanical S-matrix can be explicitly calculated by the method first introduced by J.B. McGuire[5] and later developed by C.N. Yang as[6,7,8]

$$\begin{aligned} S_{2N} &= {}^{out}\langle (N+1,\ldots,2N)(1,2,\ldots,N) | (1,2,\ldots,N)(N+1,\ldots,2N)\rangle^{in} \\ &= \prod_{j=1}^{N-1} \left(\frac{ia-j}{ia+j}\right)^2 \left(\frac{ia-N}{ia+N}\right) \\ &= \frac{ia-N}{ia+N} \left\{\frac{\Gamma(ia)\Gamma(N-ia)}{\Gamma(-ia)\Gamma(N+ia)}\right\}^2 \quad , \end{aligned} \tag{10}$$

It has been shown that, for large N values, the quantum many body expressions tend to the classical ones[9]. For example, the Bethe ansatz for N-particle bound state wavefunction tends to have the surface of envelope given by the classical one-soliton solution. In addition, the time-delay calculated from the S-matrix Eq. (10), gives rise to

$$\Delta t = \frac{\hbar}{i} \frac{\partial}{\partial E} \ln S_{2N}$$

$$= - \hbar \left\{ \frac{2N}{N^2+a^2} + 4[-\mathrm{Re}\psi(ia) + \mathrm{Re}\psi(N-ia)] \right\} \frac{\partial a}{\partial E}$$

$$= - \frac{4\hbar^3}{m\varepsilon^4} \frac{1}{aN} \left\{ \ln(1 + \frac{N^2}{a^2}) + O(\frac{1}{N-ia})^2 + O(\frac{1}{a})^2 \right\} \qquad (11)$$

which coincides with Eq. (6) to the leading order in a and N. Thus the classical solution of soliton scattering can be regarded as the collective motion of two N-particle bound states. Yoon and Negele[9] discussed the problem in the framework of the time dependent Hartree (TDH) approximation.

In a phenomenological treatment of collective motion of a quantum mechanical many body system, it is often introduced an effective Lagrangean (or Hamiltonian) for the collective variables in interest. In such a treatment, the effective Hamiltonian is usually obtained starting from the expectation value of the energy of the system under the state for which collective variables are suitably introduced. Schematically,

$$H_{eff}(Q,P) = \langle\psi_{Q,P}|H|\psi_{Q,P}\rangle$$

where $|\psi_{Q,P}\rangle$ is the statevector of the system parametrized by the collective canonical variables Q and P.

Once the effective Hamiltonian is defined as a function of Q and P, one may expect that the quantum mechanics of the collective motion can be described by the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} \phi(Q,t) = \hat{H}_{eff}\, \phi(Q,t) \qquad (13)$$

where $\phi(Q,t)$ is the wavefunction for the collective variable Q, and $\hat{H}_{eff}$ is the Hamiltonian operator obtained by the usual quantization procedure.

Although very intuitive and frequently used for the study of collective phenomena, especially in Nuclear Physics, its theoretical background is not justified nor clear. The more rigorous treatment of collective motion should be found in the mean-field theory like the TDHF or the Green function formulation.

One of the crucial point of such a phenomenological procedure is that the Hamiltonian is defined as the expectation value of the energy of the system. This averaging process usually washes out the quantum mechanical information of the many body character, so that it is hardly recuperated by the "requantization" procedure. In particular it may happen that the quantum dynamics described by such a requantized Hamiltonian has nothing to do with the real quantum mechanical property of the original system.

The NSE is the best to study this problem, since the exact solution is known. We may proceed the phenomenological quantization procedure on the NSE and compare the result with the exact one.

In order to find the effective Hamiltonian which describes the collective motion (in our case, the motion of the distance of two centers of mass of solitons), we start with the expression of the time-delay, Eq. (6).

Writing the classical Hamiltonian as

$$H_{cl} = \frac{\mu}{2}\,\dot{x}^2 + U_{eff}(x) \quad , \tag{14}$$

where $\mu = \frac{1}{2}\,M = \frac{1}{2}\,mN$ is the reduced mass, and $x = x_1 - x_2$, U_{eff} the effective potential, the time-delay is expressed as

$$\Delta t = \sqrt{\frac{M}{4}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{E}} - \frac{1}{\sqrt{E-U_{eff}}}\right) dx \tag{15}$$

Expanding Eq. (15) with respect to $\frac{1}{E}$, and comparing the result with the expansion of Eq. (6), we find[7]

$$U_{eff}(x) = -\eta N^2\varepsilon^2/\cosh^2(\eta x) \tag{16}$$

According to the phenomenological quantization scheme, we consider the following Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\phi(x,t) = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2}\phi(x,t) + U_{eff}(x)\phi(x,t) \tag{17}$$

We then like to see whether this equation describes the quantum mechanical aspects of the collective variable x corresponding to the original many body system, Eq. (2). For this purpose, it is convenient to calculate the S matrix corresponding to Eq. (17). Eq. (17) is known as exactly soluble one, and the symmetrized S-matrix (= T+R; T: transmission coefficient, R: reflection coefficient) is given by[10]

$$S_{eff} = \frac{\Gamma(2ia)\Gamma(2\lambda-2ia)}{\Gamma(-2ia)\Gamma(2\lambda+2ia)}\;\frac{\cos\pi(\lambda-ia)}{\cos\pi(\lambda+ia)} \tag{18}$$

where $\lambda = \frac{1}{4}+\frac{1}{4}\sqrt{1+16N^2}$, with a and N are given by Eqs. (7) and (8).

To compare this result with Eq. (10), we first make use of the formula[11],

$$\Gamma(2Z) = (2\pi)^{-1/2}\,2^{2Z-\frac{1}{2}}\,\Gamma(Z)\Gamma(Z+\tfrac{1}{2}) \tag{19}$$

and rewrite Eq. (18) as

$$S_{eff} = \frac{\Gamma(ia)\Gamma(ia+\frac{1}{2})\Gamma(\lambda-ia)\Gamma(\lambda-ia+\frac{1}{2})\cos\pi(\lambda-ia)}{\Gamma(-ia)\Gamma(-ia+\frac{1}{2})\Gamma(\lambda+ia)\Gamma(\lambda+ia+\frac{1}{2})\cos\pi(\lambda+ia)} \tag{20}$$

Noting that

$$\lambda = N + \frac{1}{4} + O(\frac{1}{N})$$

and with the help of the formula[11]

$$\Gamma(Z+\alpha) = Z^{\alpha}\Gamma(Z)[1 + \frac{\alpha(\alpha-1)}{2Z} + O(\frac{1}{Z^2})] \tag{21}$$

we find

$$S_{eff} = \frac{-(N-ia)}{(N+ia)}\left\{\frac{\Gamma(ia)\Gamma(N-ia)}{\Gamma(-ia)\Gamma(N+ia)}\right\}^2\{1 + O(\frac{1}{ia}) + O(\frac{1}{N})\} \tag{22}$$

Thus we see that our S_{eff} tends to the exact S-matrix S_{2N} of Eq. (10) for large a and N. However, this is not yet so satisfactory since such information from S_{eff} in this domain of large a and N is exactly the same as that of the classical quantity Δt, Eq.(6).

To examine the quantum effect reflected in Eq. (18), we look for bound state poles of S_{eff}. These bound state poles come from positive zeros of $\cos\pi(\lambda+ia)$, i.e.,

$$-ia_{\ell} = (\lambda - \ell - \frac{1}{2}) \quad , \quad \ell \geq 0 \tag{23}$$

The ground state $\ell = 0$ energy is then given by

$$\begin{aligned} E_{g.s.} &= -(\lambda - \frac{1}{2})^2 \frac{mN\varepsilon^4}{4\hbar^2} \\ &= -\frac{m\varepsilon^4 N^3}{4\hbar^2}\{1 + O(\frac{1}{N})\} \quad , \end{aligned} \tag{24}$$

which should be compared to the exact value

$$E_{exact} = - \frac{m\varepsilon^4}{4\hbar^2} N^3 . \tag{25}$$

It is interesting to observe that for N >> 1, our phenomenological quantization scheme gives an excellent estimate to the exact value of the binding energy of the original quantum mechanical 2N-body system. This fact and Eq. (22) seem to support the adequacy of the phenomenological quantization.

However there exists a crucially week point in claiming that Eq. (17) describes correctly the quantum mechanical property of the collective motion. That is, our S_{eff} exhibits too many bound state poles[10], whereas there is only one for the exact expression. These extra poles spoil essentially the scattering amplitude in the energy region where quantum effects are important.

As a conclusion, we summarize our present investigation as follows:

1) We calculated the effective potential between two identical solitons of NSE starting from the classical expression for time--delay.
2) Using this effective potential, we studied the quantum mechanics of two solitons, according to the phenomenological quantization scheme.
3) It is shown that for large N and a values (large E or weak coupling) our effective S-matrix reproduces exact S-matriz of the original 2N system.
4) The ground state binding energy is correctly estimated for large N.
5) Scattering amplitude for low energy does not have correct behavior even for large N.

To proceed the phenomenological quantization scheme, we took the classical expression for the time delay Eq. (6) as the quantum expectation value. In order to follow exactly the spirit of the method, Eq. (11) may be better than Eq. (6) to calculate U_{eff}. However, we expect that conclusion here stay unaltered.

ACKNOWLEDGMENT

The authors are grateful to Drs. J. Neto, M.C. Nemes and A.F.R. Toledo Piza for interesting and stimulus discussion.

REFERENCES

1. See for example, Alwyn C. Scott, F.Y.F. Chu, and David W. McLaughlin, Proceedings of the IEEE 61, 1443 (1973).

2. See for example, D.J. Kaup, J. Math. Phys. 16,2036(1975), L.D. Faddeev, Topics in Current Physics, Solitons (Spring-Verlag, 1980) p. 339, Dennis B. Greamer et. al., Phys. Rev. D 21, 1523 (1980), Mamoru Uchinami et. al., Prog. Theor. Phys. 66,1284(1981).

3. V.E. Zakharov and A.B. Shabat, Zh. Eksp. Teor. Fiz. 61, 118 (1971) [Sov. Phys. - JETP 34,62(1972)].

4. See Ref. 2, D.J. Kaup.

5. J.B. McGuire, J. Math. Phys. 5,622(1964).

6. C.N. Yang, Phys. Rev. 168,1920(1968).

7. J.N. Maki, Thesis, Centro Brasileiro de Pesquisas Físicas, in preparation.

8. B. Yoon and J.W. Negele, Phys. Rev. A 16,1451(1977).

9. M. Wadati and M. Sakagami, J. Phys. Soc. Japan 53,1933(1984), M. Wadati, A. Kuniba, and T. Konishi, J. Phys. Soc. Japan 54, 1710(1985).

10. See for example J.N. Ginocchio, Ann. Phys. 152,203(1984), S. Flügge, Practical Quantum Mechanics, (Springer-Verlag, 1971) p. 94.

11. M. Abramowitz and J.A. Stegun, Handbook of Mathematical Functions (Dover Publications, Inc., New York) p. 256 and p. 257.