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SZEKERES SPACETIMES WITH HEAT FLOW

by

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ABSTRACT

We give a new class of inhomogeneous exact solutions of Einstein field equations. They generalize the dust-filled models found by Szekeres. In the course of time a subclass of models present a friedmannian phase.

Key words: Cosmology; Szekeres models; General relativity; Heat flow.

In the context of classical cosmology, it is important to obtain inhomogeneous exact solutions of the Einstein field equations for imperfect fluid as source of curvature, for several reasons. One the most interesting of them is related to the present entropy of the universe. As is well known, the rate of entropy produced by nonadiabatic mechanisms in an initially homogeneous background seems to be insufficient to account for the high entropy per particle of the universe [1, 2]. So, it is often believed that mechanisms, such as shear viscosity or heat conduction in the early inhomogeneous universe, can generate all necessary entropy and also be responsible by the high regularity in the structure of the universe at the cosmological scale nowadays [3]. This question has a close correlation with the program of chaotic cosmology [4, 5].

On the other hand, there are few inhomogeneous exact solutions that can be used as realistic universe models. A remarkable exception is the family of dust-filled solutions found by Szekeres [6]. Several theoretical questions have been examined in this background [7,8,9,10]. These solutions are divided in two classes usually denoted by classes I and II. The models of the first class generalize the Tolman-Bondi solutions and are useful to study nonsymmetrical gravitational colapse [7]. Those of the class II are more important as cosmological models, because these can closely approximate, over a finite time interval, the FRW dust models [8]. In a recent paper [11], this last result has been generalized for a fluid in non thermal equilibrium state. Actually only the behaviour of one particular model of class II was investigated. The FRW models with euclidean sections were obtained in the limit of a large cosmological time. Here, we show that it is possible to include the hyperbolic and closed FRW models too.

Consider now the line element of Szekeres' cosmological models class-II [8]. (in our units $8\pi G = c = 1$) :

$$ds^2 = dt^2 - Q^2dx^2 - R^2(dy^2 + h^2dz^2) \; , \tag{1}$$

where

$$Q = Q(t, x, y, z) \quad , \quad R = R(t) \quad \text{and} \quad h = h(y) \; .$$

The Einstein field equations for a fluid with heat flow are

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}R = - [(\rho + p)V_\alpha V_\beta - pg_{\alpha\beta} + q_\alpha V_\beta + q_\beta V_\alpha], \tag{2}$$

where V_α, ρ and p are the four velocity of matter, the mass-energy density and the pressure respectively; q_α is the heat flow and satisfies the equation $q_\alpha V^\alpha = 0$.

In the comoving coordinates system ($V^\alpha = \delta^\alpha_o$), the nontrivial Einstein equations ($\cdot \equiv \partial/\partial t$ and $Q_{,i} \equiv \partial Q/\partial x^i$ $i = 1, 2, 3 \equiv x, y, z$) are

$$QR^2\rho = Q\dot{R}^2 + 2R\dot{Q}\dot{R} - Q_{,22} - h^{-2}(Q_{,33} + hh_{,2}Q_{,2} + hh_{,22}Q) \; , \tag{3}$$

$$R^2p = -2R\ddot{R} - \dot{R}^2 + h_{,22}\, h^{-1} \; , \tag{4}$$

$$QRp = -Q\ddot{R} - \dot{Q}\dot{R} - \ddot{Q}R + h^{-2}R^{-1}\, (Q_{,33} + hh_{,2}Q_{,2}) \; , \tag{5}$$

$$h^2QRp = -Q\ddot{R} - \dot{Q}\dot{R} - \ddot{Q}R + R^{-1}Q_{,22} \; , \tag{6}$$

$$q_1 = 0 \; , \tag{7}$$

$$Qq_2 = Q_{,2}R^{-1}\dot{R} - \dot{Q}_{,2} \; , \tag{8}$$

$$Qq_3 = Q_{,3}R^{-1}\dot{R} - \dot{Q}_{,3} \quad , \tag{9}$$

$$0 = Q_{,3}h^{-1}h_{,2} - Q_{,23} \quad . \tag{10}$$

From Eq.(7) and condition $q_\alpha V^\alpha = 0$, the heat flow is restricted to $q_\mu = (0, 0, q_2, q_3)$. When the pressure is zero, these equations are easily integrated. The mass energy density and the heat flow are defined by Eqs.(3), (8) and (9), whereas the remainder equations give us the metrical components. In this case, solving (4) we obtain

$$2R\ddot{R} + \dot{R}^2 + \varepsilon = 0 \quad , \tag{11}$$

and

$$h_{,22}h^{-1} = -\varepsilon \quad . \tag{12}$$

Then R satisfies the standard equation of FRW models and $\varepsilon = 0, \pm 1$ is the curvature constant of the bidimensional section $t \equiv$ const, x = const [6, 8].

For all values of ε, the Q function is given by

$$Q = AR + BM + T \tag{13}$$

where

$$A = A(x, y, z) \; , \; B = B(x, y, z) \; , \; M = M(t) \; \text{and} \; T = T(x, t) \; .$$

The models can be classified according to the values of ε. The mass energy density, the components of heat flow and the functions A, R, B, M and T will be given below in each case.

In what follows σ, ν, η, α, γ, δ, β and μ denote arbitrary functions of x. R_o is a positive constant and ω is the usual parameter defined by $dt = Rd\omega$. We will use the notation of Ref.[8].

Elliptic Model $\varepsilon = +1$, $h = \sin y$:

$$A = (\sigma \cos z + \nu \sin z) \sin y + \eta \cos y \quad ,$$

$$B = (\alpha \cos z + \gamma \sin z) \sin y + \delta \cos y \quad ,$$

$$R = 2R_o \sin^2\frac{\omega}{2} \quad , \quad t = R_o(\omega - \sin\omega) \quad ,$$

$$M = R_o \cot\frac{\omega}{2} (1 + 2\sin^2\frac{\omega}{2}) \quad ,$$

$$T = \beta(\frac{\omega}{2} \cot\frac{\omega}{2} - 1) + \mu \cot\frac{\omega}{2} \quad ,$$

$$\rho = \frac{6R_oA - \beta + 6R_oB \cot\frac{\omega}{2}}{QR^2} \quad , \tag{14}$$

$$q_2 = \frac{3R_o}{QR^2} [(\alpha \cos z + \gamma \sin z) \cos y - \delta \sin y] \quad , \tag{15}$$

$$q_3 = \frac{3R_o}{QR^2} [(\gamma \cos z - \alpha \sin z) \sin y] \quad . \tag{16}$$

Hyperbolic Model $\varepsilon = -1$, $h = \cosh y$:

$$A = (\sigma \cosh z + \nu \sinh z) \cosh y + \eta \sinh y \ ,$$

$$B = (\alpha \cosh z + \gamma \sinh z) \cosh y + \delta \sinh y \ ,$$

$$R = 2R_o \sinh^2\frac{\omega}{2} \quad , \quad t = R_o(\sinh\omega - \omega) \ ,$$

$$M = R_o \coth\frac{\omega}{2} (1 - 2\sinh^2\frac{\omega}{2}) ,$$

$$T = \beta(\frac{\omega}{2} \coth\frac{\omega}{2} - 1) + \mu \coth\frac{\omega}{2} \quad ,$$

$$\rho = \frac{6R_o A + \beta - 6R_o B \coth \frac{\omega}{2}}{QR^2} , \tag{17}$$

$$q_2 = \frac{3R_o}{QR^2} [(\alpha \cosh z + \gamma \sinh z) \sinh y + \delta \cosh y] , \tag{18}$$

$$q_3 = \frac{3R_o}{QR^2} [(\alpha \sinh z + \gamma \cosh z) \cosh y] . \tag{19}$$

For completeness we list the case $\varepsilon = 0$ presented in the Ref.[11], in this notation.

Parabolic Model $\varepsilon = 0$, $h = 1$:

$$A = \beta(y^2 + z^2) + \sigma y + \nu z + \eta ,$$

$$B = \alpha(y^2 + z^2) + \gamma y + \delta z ,$$

$$R = t^{2/3} ,$$

$$M = t^{-1/3} ,$$

$$T = \frac{9}{5} \beta t^{4/3} + \mu t^{-1/3} ,$$

$$\rho = \frac{4A - 12\alpha R^{-1/2}}{3QR^2} , \tag{20}$$

$$q_2 = \frac{2\alpha y + \gamma}{QR^2} , \tag{21}$$

$$q_3 = \frac{2\alpha z + \delta}{QR^2} . \tag{22}$$

To see the relation between our models and Szekeres solutions we must take $B = 0$ in all cases. The resulting solutions are the corresponding models of Szekeres class-II [8]. The asymptotic behaviour of physical quantities can be seen by retaining only the leading terms in the variable t for $\varepsilon = 0$, or in the variable ω for $\varepsilon = \pm 1$, in all expressions. For dust filled solutions ($B = 0$), this question was studied by Bonnor and Tomimura [8]. In our case ($B \neq 0$), the evolution depends also on the arbitrary functions. In general, these solutions start as inhomogeneous and anisotropic models and remain inhomogeneous and anisotropic. However, if the arbitrary functions are suitably restricted, a Friedmannian era can be obtained in certain limits. For $\varepsilon = \pm 1$ it is sufficient to take $\beta(x) = 0$, and for $\varepsilon = 0$ we have taken $\beta(x) = \alpha(x) = 0$ (see Ref.[11]). In the hyperbolic case the FRW phase occurs in the limit of large cosmological time ($\omega >> 1$), whereas in the closed model we must take the limit where the parametric coordinate ω tends to the value π i.e., when the bidimensional section t = const, x = const has maximum radius. Actually one can see from Eqs.(14)-(16) and (17)-(19), that the intensity of heat flow $q = (-q^{\alpha}q_{\alpha})^{1/2}$ decreases to zero faster than the density in these limits. It seems that this FRW phase for closed models was not suspected in the Ref.[8]. Such as in the "flat case" [11], it can be easily shown that the spacetimes of our models are Petrov I type (algebraically general), for nonzero heat flow. But when the heat flow is zero, the models are Petrov D type. For their minor importance in the cosmological problem, the other models of class-II were not generalized here. Up to now, pressure effects were not considered, but our experience with the "flat case" strongly suggests that our results will not be considerably modified if isotropic pressure is added. We intend to examine at length the evolution, pressure effects and the thermodynamics of the models in a **forthcoming communication**.

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