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WILSON LOOPS IN KERR GRAVITATION

by

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Abstract

The ordered integrals for several paths in Kerr gravitation is computed in a compact form. When the path is closed we discuss its relation with the angular parallel displacement and calculate the corresponding Wilson loop. The validity of Mandelstam relations for gauge fields is also explicitly verified.

The Wilson loop [1] [2] has arisen considerable attention in gauge theories because it is a gauge invariant quantity and it is thought that it can act as a dynamical variable with its own evolution equations involving their variational derivatives [3] [4] [5] [6] [7]. It should then be an important variable for the quantization of non-abelian gauge theories. If this is the case, it seems interesting to compute the value of the Wilson loop(WL)(and related strings) for some classical configurations of the gauge fields. This could also help for an understanding of an eventual classical theory based on WL.

In this sense, in a previous paper, we presented some results of an actual computation [8] of WL for the instanton [9]. It seems of interest to perform similar calculations for some configurations of the gravitational field. The WL is related to the parallel displacement of a vector along a closed path, which in the gravitational case has a space-time geometrical meaning. We shall perform the computations for the gravitational field corresponding to the Kerr-metric [10] which contains as a particular case, the Schwarzschild metric [11].

We shall call

$$U(c) = P\, e^{\int_c \Gamma_\mu dx^\mu} \tag{1}$$

where P means ordered product along the curve c, Γ_μ is the tetradic connection which, for the Kerr metric is given explicitly by [12]

$$\Gamma^4_{\mu 2}\, dx^\mu = -a \sin\theta\cos\theta \frac{\Delta^{1/2}}{\Sigma}\, d\phi = \Gamma^2_{\mu 4}\, dx^\mu$$

$$\Gamma^4_{\mu 3}\, dx^\mu = a \cos\theta \frac{\Delta^{1/2}}{\Sigma}\, d\theta - \frac{a\, r\sin\theta}{\Sigma\, \Delta^{1/2}}\, dr = \Gamma^3_{\mu 4}\, dx^\mu$$

$$\Gamma^4_{\mu 1}\, dx^\mu = \frac{m}{\Sigma^2}(r^2 - a^2\cos^2\theta)dt + \frac{a\sin^2\theta}{\Sigma^2}\left[(m-r)\Sigma - 2mr^2\right]d\phi = \Gamma^1_{\mu 4}\, dx^\mu \tag{2}$$

$$\Gamma^2_{\mu 3}\, dx^\mu = \frac{2mr}{\Sigma^2}\, a\cos\theta\, dt - \frac{\cos\theta}{\Sigma^2}\left[\Sigma(r^2+a^2) + 2mra^2\sin^2\theta\right]dr = -\Gamma^3_{\mu 2}\, dx^\mu$$

$$\Gamma^3_{\mu 1}\, dx^\mu = r\sin\theta \frac{\Delta^{1/2}}{\Sigma}\, d\phi = -\Gamma^1_{\mu 3}\, dx^\mu$$

$$\Gamma^2_{\mu 1}\, dx^\mu = \frac{r\Delta^{1/2}}{\Sigma}\, d\theta + \frac{a^2\sin\theta\cos\theta}{\Sigma\Delta^{1/2}}\, dr = -\Gamma^1_{\mu 2}\, dx$$

(outside the "ergosphere") where

$\Sigma = r^2 + a^2\cos^2\theta$ (with the diagonal Lorentz metric $\eta = (1,1,1,-1)$) (3)

$\Delta = r^2 - 2mr + a^2$ (4)

a is the angular momentum per unit mass

The WL corresponding to a closed curve c will be

$$W(c) = \mathrm{Tr}\; U(c) \tag{5}$$

We shall first consider circles with center at the origin with fixed values of r, θ, t. So, in this case

$$\Gamma_\mu dx^\mu = \Gamma_\phi\, d\phi \quad \text{with} \quad dr = d\theta = dt = 0 \tag{6}$$

and from (2).

$$\Gamma_\phi = \begin{pmatrix} 0 & 0 & A & B \\ 0 & 0 & C & D \\ -A & -C & 0 & 0 \\ B & D & 0 & 0 \end{pmatrix} \qquad \begin{aligned} A &= -\frac{r\sin\theta\ \Delta^{1/2}}{\Sigma} \\ B &= \frac{a\sin^2\theta}{\Sigma^2}\left[(m-r)\Sigma-2mr^2\right] \\ C &= \frac{\cos\theta}{\Sigma^2}\left[(r^2+a^2)\Sigma+2mra^2\sin^2\theta\right| r^2 \\ D &= -\frac{\Delta^{1/2}}{\Sigma}\, a\sin\theta\cos\theta \end{aligned} \tag{7}$$

As Γ_ϕ is independent of ϕ

$$U = P e^{\int_{\phi_1}^{\phi_2} \Gamma_\phi d\phi} = e^{\Gamma_\phi(\phi_2-\phi_1)} \tag{8}$$

In particular, for $\theta=\frac{\pi}{2}$ the elements of the matrix (7) are given by

$$A = -\frac{\Delta^{1/2}}{r}\ ; \quad B = -\frac{a}{r}\left(1+\frac{m}{r}\right). \tag{9}$$

Observe that, for $a=0$ and $r=2m$ (Schwarzschild radius) we have $\Gamma_\phi \equiv 0$

From (7) and (9) it is easy to see that

$$\Gamma_\phi^3 = -\frac{1}{r^2}\left[\Delta-a^2\left(1+\frac{m}{r}\right)^2\right]\Gamma_\phi \equiv -A_\phi^2\Gamma_\phi \text{(Definition of } A_\phi) \tag{10}$$

This relation implies that

$$e^{\Gamma_\phi(\phi_2-\phi_1)} = 1+\frac{\Gamma_\phi}{A_\phi}\sin A_\phi(\phi_2-\phi_1)+\frac{\Gamma_\phi^2}{A_\phi^2}\left(1-\cos A_\phi(\phi_2-\phi_1)\right) \tag{11}$$

and, taking the trace for a complete circle

$$W = \mathrm{Tr}\ e^{2\pi\Gamma_\phi} = 2(1+\cos 2\pi A_\phi) \tag{12}$$

note that $\mathrm{Tr}\ \Gamma_\phi^2=-2A_\phi^2$

Let us compute (1) for a curve $r(s)$, $\theta(s)$ contained in a meridian plane, we need

$$\Gamma_s ds = (\Gamma_\theta \dot{\theta} + \Gamma_r \dot{r}) ds \tag{13}$$

where, from (2)

$$\Gamma_s = \begin{pmatrix} f\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & 0 \\ 0 & g\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} \qquad \begin{aligned} f &= \frac{r\Delta^{1/2}}{\Sigma}\dot{\theta} + \frac{a^2 \sin\theta\cos\theta\ \dot{r}}{\Sigma\ \Delta^{1/2}} \\ g &= a\cos\theta\ \frac{\Delta^{1/2}}{\Sigma}\dot{\theta} - \frac{a r \sin\theta\ \dot{r}}{\Sigma\ \Delta^{1/2}} \end{aligned} \tag{14}$$

Then

$$P e^{\int \Gamma_s ds} = e^{\int \Gamma_s ds} , \tag{15}$$

as the matrices commute for different values of s. In particular we see that for the Schwarzschild metric, due to the spherical symmetry, the property (15) should hold for any curve contained in an arbitrary plane through the origin (at least, for a convenient gauge).

In particular, for a meridian circle, $\dot{r}=0$ $\dot{\theta}=1$ we obtain

$$e^{2\int \Gamma_\theta d\theta} = \begin{pmatrix} e^{\frac{2\pi\Delta^{1/2}}{\sqrt{r^2+a^2}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \tag{16}$$

From here

$$W = 2(1+\cos 2\pi\ A_\theta) \tag{17}$$

with

$$A_\theta = \frac{\sqrt{r^2-2mr+a^2}}{\sqrt{r^2+a^2}} \tag{18}$$

note that for $a=0$ $A_\theta=A_\phi$, as it should.

For a radial segment we obtain form (14) with $\dot\theta=0$ $\dot r=1$

$$e^{\int_{r_1}^{r_2}\Gamma_r dr} = \begin{pmatrix} e^{f'}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} & 0 \\ 0 & e^{g'}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\end{pmatrix} \tag{19}$$

with

$$f' = a\sin\theta\cos\theta\int_{r_1}^{r_2}\frac{dr}{\Sigma\Delta^{1/2}} \qquad g' = -a\sin\theta\int_{r_1}^{r_2}\frac{r}{\Sigma\Delta^{1/2}}\,dr$$

note that in Schwarzschild case ($a=0$) we have $\Gamma_r=0$ $U=1$

For a translation in time

$\Gamma_\mu dx^\mu = \Gamma_t dt$, with Γ_t (from (2)) being

$$\Gamma_t = \alpha\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix} + \beta\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix} = \alpha\Gamma' + \beta\Gamma''$$

with

$$\alpha = \frac{m}{\Sigma^2}(r^2-a^2\cos^2\theta) \qquad \beta = \frac{2mr}{\Sigma^2}\,a\cos\theta$$

So that

$$U= e^{\Gamma_t(t_2-t_1)} = e^{\Gamma_t\tau} = e^{(\alpha\Gamma'+\beta\Gamma'')\tau} = e^{\Gamma'\alpha\tau} \; e^{\Gamma''\beta\tau}$$

$$U= \Gamma'\mathrm{Sh}\;\alpha\tau + \Gamma''\sin\beta\tau + \Gamma'^2\;\mathrm{Ch}\;\alpha\tau - \Gamma''^2\cos\beta\tau \tag{20}$$

If we consider α=0 and a closed path formed by two radial segments (which contribute with a unit factor) and two temporal segments at radius r_1 and r_2 (resp.)the result is:

$$U = 1+\Gamma'\mathrm{Sh}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)m\tau + \Gamma'^2\left[\mathrm{Ch}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)m\tau - 1\right] \tag{21}$$

from which, the WL is:

$$W = 2\left[1+\mathrm{Ch}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)m\tau\right] \quad (\mathrm{cf}(12)\ \mathrm{and}\ (17))$$

For $r_1 \simeq r_2 = r >> m$ (21) reduces to

$$U = 1 + \Gamma'\,\frac{2m(r_2-r_1)}{r^3} + \Gamma'^2\,\frac{2m^2(r_2-r_1)^2}{r^6} + \ldots$$

To compute the parallel displacement of a vector A^a (tetradic index) we simply take its product with the U-matrix

$$A'^a = U^a{}_b A^b \qquad A'=UA \tag{23}$$

When U corresponds to a closed path, starting and ending at the point where A is defined, we can take the scalar product between A' and A, thus defining an "angle" of the parallel displacement along that path. It is easy to check that U is unitary in the sense that

$A'.A' = A.A.$ ie $A'^a \eta_{ab} A'^b = A^a \eta_{ab} A^b$ or, in matrix notation

$$A'\eta A = A\eta A$$

It is straightforward to show that this implies

$$\eta\tilde{U}\eta U = 1 \qquad (24)$$

where $\tilde{U}$ means the transpose of U. Further, as $U=e^V$ it is easy to see that if

$$\eta\tilde{V}\eta = -V \qquad (25)$$

then (24) is satisfied.

When neither A nor A' has temporal component, one can define a real angle χ by means of the equation

$$\cos\chi = \frac{A'\eta A}{A.A} = \frac{A\eta U^{-1} A}{A.A} = \frac{A(\tilde{U}\eta)A}{A.A} = \frac{A\eta UA}{A.A} \qquad (26)$$

From (23) it is straightforward to see (by choosing a vector $A_{(a)}$ with component $A^b_{(a)} = \delta^b_a$ that

$$U^b_a = A'^b_{(a)}$$

ie, the elements of U are the components of the parallel translated vector, pointing originally in the (a) direction.

From (26) and (27) it follows that, if $A'_{(a)}$ is a space vector, then, the corresponding diagonal element of U is the cosine of the angle between the two vectors.

$$U^a{}_a = \cos \chi_a \tag{28}$$

If we consider a circle in the equator and the index a=1 then eq(11) gives.

$$\cos\chi_1 = \cos 2\pi A_\phi$$

$$|\chi_1| = |2\pi A_\phi + 2\pi n|$$

As for $m\to 0$ we must have $\chi_1 \to 0$ we choose n=-1 so

$$|\chi_1| = 2\pi |A_\phi - 1| \tag{29}$$

From (10) we see that the dominant term for $r\to\infty$ is simply

$$|\chi_1| = \frac{2\pi m}{r} \quad r\to\infty \tag{30}$$

If in (29) we take $a=0$ we have the angle corresponding to the Schwarzschild metric

$$|\chi^s| = 2\pi |A_\phi(a=0) - 1| \tag{31}$$

Note that for the Schwarzschild radius we have $A_\phi = 0$ and $\chi_1^s = 2\pi$.

For "a" small, the Kerr metric introduces a correction which can be computed from (10) with the results

$$\chi_1 \simeq \chi_1^{(s)} - 2\pi a^2 \frac{m}{r^3} \quad (\frac{m}{r} \ll 1) \tag{32}$$

In general, when we come back to the starting point of the closed curve, the parallel transferred vector and the original one are related by

$$A' = A + \alpha \tag{33}$$

but, as both have the same norm

$$A' \circ A' = A.A + \alpha.\alpha + 2\alpha.A = A \circ A$$

so

$$\alpha.A = -\frac{1}{2}\,\alpha.\alpha.$$

then

$$A'.A = A.A + \alpha.A = 1 - \frac{1}{2}\,\alpha \circ \alpha \qquad (A \circ A = 1)$$

From this and (27), we get for the Wilson loop

$$W = 4 - \frac{1}{2}\,\alpha_{(a)} \cdot \alpha_{(b)}\,\eta^{ab} \tag{34}$$

Obviously, W=4 is the "vacuum" (or flat) value for the Wilson loop in gravitation, while the diference (4-W) is a measure, as can be seen from (34) of the Minkoskian modulus of A'-A

As an example, from (12) and (30) (or (17), (18))

$$W = 4 - \left(\frac{2\pi m}{r}\right)^2 \qquad \text{for } r \to \infty \tag{35}$$

We should note, however, that W=4 is a necessary but not sufficient condition for the "vacuum" state, (for r=2m, Schwarzschild radius, W=4) as can be seen from (10) and (12) , and the curvature is not zero as is well known. In fact, we shall see that the curvature is related to the derivatives of the Wilson loop.

From the expressions for the curvature tensor in Schwarzschild metric

$$R^{a}_{13_b} = - \frac{m \operatorname{sen}\theta}{r^2\sqrt{1-2m/r}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (36)$$

$$R^{a}_{23_b} = \frac{2m \operatorname{sen}\theta}{r} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (37)$$

(a,b, tetradic indices) and the expression (10) for $\Delta\phi=2\pi$ and "a"=0 one can easily check the relations

$$\frac{\partial W}{\partial r} = \int_0^{2\pi} d4 \ \mathrm{Tr}(R_{13}U) = 2\pi \ \mathrm{Tr}(R_{13}U) \qquad (38)$$

$$\frac{\partial W}{\partial \theta} = \int_0^{2\pi} d4 \ \mathrm{Tr}(R_{23}U) = 2\pi \ \mathrm{Tr}(R_{23}U) \qquad (39)$$

where the left hand side can be directly computed from (12)

Forms (38), (39) are particular cases of

$$\frac{\partial W}{\partial x_\nu} = \oint ds \quad T\gamma\{R_{\nu\mu}U\} \frac{dy^\mu}{ds} \tag{40}$$

which can be deduced from the general Mandelstam relation[2] [8].

We can also try to see what happens with the WL inside the Schwarzschild sphere (for "a"=0). For simplicity we consider $\Theta = \frac{\pi}{2}$ with this aim we can use Kruskaal coordinates by means of which we can calculate (8). The result so obtained is equivalent to the one we get when we take expressions in Schwarschild solution with the following prescription:

for A_ϕ we take

$$A_\phi = \sqrt{|1-2m/r|}$$

and for Γ_ϕ

$$\Gamma_\phi = \sqrt{|1-2m/r|} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Recalling that r and t interchange roles inside the Schwarzschild radius, we note that now 1 is a temporal index, while 4 is spacial and as a result $\Gamma^a_b = -\Gamma^b_a$ when neither a nor b is 1 and $\Gamma^a_b = \Gamma^b_a$ when either "a" or "b" is 1. As a result the U matrix contains hyperbolic functions when for r>2m it contained natural trigonometric functions then, instead of (11) for r<2m

$$U = 1 + \frac{\Gamma\phi}{A\phi} \,\mathrm{Sh}\, 2\pi A_\phi - \frac{\Gamma^2\phi}{A^2_\phi} (1 - \mathrm{Ch}\, 2\pi A_\phi)$$

In all previous considerations the U matrices (Γ matrices)are refered to tetradic coordinates. If one wants to go to tensor indices it is necessary to use the mixed matrix:

$$h^{a}{}_{\alpha} = \begin{pmatrix} \left(\frac{\Sigma}{\Delta}\right)^{1/2} & 0 & 0 & 0 \\ 0 & \Sigma^{1/2} & 0 & 0 \\ 0 & 0 & \frac{(r^2+a^2)\sin\theta}{\Sigma^{1/2}} & -\frac{a\sin\theta}{\Sigma^{1/2}} \\ 0 & 0 & -a\sin^2\theta\left(\frac{\Delta}{\Sigma}\right)^{1/2} & \left(\frac{\Delta}{\Sigma}\right)^{1/2} \end{pmatrix} \tag{43}$$

(and its inverse= h^{α}_{a})

A U matrix transforms as follows

$$\overline{U}^{\alpha}_{\beta} = h^{\alpha}_{a}\, U^{a}_{b}\, h^{b}_{\beta} \tag{44}$$

While the Wilson loop, being a trace of a closed path, it is invariant:

$$W = \mathrm{Tr}U = U^{a}_{a} = \overline{U}^{\alpha}_{\alpha} \tag{45}$$

For the Γ matrices the transformation to tensor indices acts like a gauge transformation so that the combination

$$h^{\alpha}_{a}\, \Gamma^{a}_{\mu b}\, h^{b}_{\beta} + h^{\alpha}_{a}\, \partial_{\mu}\, h^{a}_{\beta} = \left\{ {\alpha \atop \mu\beta} \right\} \tag{46}$$

is just a Christoffel symbol

Note that the fact that the ordered exponential integrals, which we have examined, are abelian when the Γs are referred to tetradic indices, does not necessarily means that they are abelian if computed by using the tensor connections (46).

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