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SELF-DUAL CLUSTER RENORMALIZATION GROUP APPROACH FOR THE SQUARE LATTICE ISING MODEL SPECIFIC HEAT AND MAGNETIZATION

by

H.O. Mártin* and C. Tsallis

* LABORATÓRIO DE FÍSICA TEÓRICA
DEPARTAMENTO DE FÍSICA - U.N.L.P.
C.C.Nº67 (1900) La Plata, ARGENTINA

CENTRO BRASILEIRO DE PESQUISAS FÍSICAS
AV. Wenceslau Braz, 71, fundos - Rio de Janeiro
22290 - R.J. - BRASIL

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H.O.Mártin*

Laboratorio de Física Teórica. Departamento de Física. U.N.L.P.
C.C.Nº67 (1900) La Plata, Argentina

and

C.Tsallis

Centro Brasileiro de Pesquisas Físicas/CNPq
Av. Wenceslau Braz 71, Rio de Janeiro, Brasil

Abstract:

A simple renormalization group approach based on self-dual clusters is proposed for two-dimensional nearest-neighbour $\frac{1}{2}$ - spin Ising model on the square lattice; it reproduces the exact critical point. We calculate the internal energy and the specific heat for vanishing external magnetic field, spontaneous magnetization and the thermal (y_T) and magnetic (y_H) critical exponents. The results obtained from the first four smallest cluster sizes strongly suggest the convergence towards the exact values when the cluster sizes increases. Even for the smallest cluster, where the calculation is very simple, the results are quite accurate, particularly in the neighbourhood of the critical point.

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1. Introduction

The possibility of studying the properties of an Ising model with a real space renormalization group transformation (RGT) has been extensively investigated (Niemeijer and van Leeuwen 1976, Barber 1977, van Leeuwen 1975 among others). Different RGT have been used (Nauenberg and Nienhuis 1974, Tjon 1974, Jayaprakash et al 1978) to calculate the internal energy, the specific heat and the spontaneous magnetization as functions of temperature.

The Migdal-Kadanoff approximate recursion relations for the nearest-neighbour square lattice in the particular limit of scale parameter $b \to 1$ preserve the duality symmetry of the model; therefore the internal energy and the specific heat with the *exact* critical point can be obtnained (Jayaprakash et al 1978, see the curve with $p = 1.0$ of Fig. 7 therein). Recently RGT based on self-dual clusters have been used to obtain the critical frontier of the bond-mixed model (Yeomans and Stinchcombe 1979, 1980, Tsallis and Levy 1980, Levy et al 1980, Curado et al 1981). Once more due to self-duality, the *exact* critical point is obtained for the pure Ising model.

The main purpose of the present work is to construct a simple RGT based on self-dual clusters, that allows quite accurate calculations of the thermal bahaviour of the internal energy and the specific heat for vanishing external magnetic field, and the spontaneous magnetization for the $\frac{1}{2}$ - spin Ising ferromagnet in nearest-neighbour square lattice.

In section 2 we introduce the RGT we shall use, and in section 3 we present the results obtained for different cluster sizes. Finally in section 4 we state our conclusions.

2. Calculation Method

2.1 Brief review of the formalism

Let us start from the dimensionless hamiltonian of an Ising-like spin system

$$\mathcal{H}(S) = K \sum_{<i,j>} S_i S_j + H \sum_i S_i + \dots ,$$

where $S_i = \pm 1$ is the spin variable related to the i-th site of the lattice; $<i,j>$ runs over all first neighbouring sites; K, H ... are the usual dimensionless coupling constants defined as $K = J/k_B T$; $H = \mu h/k_B T$ in terms of the exchange energy J, the magnetic field h, the magneton μ and the temperature T. Through a given RGR we obtain the renormalized hamiltonian $\mathcal{H}'(S')$

$$\mathcal{H}'(S') = K' \sum_{<i',j'>} S_{i'} S_{j'} + H' \sum_{i'} S_{i'} + \dots ,$$

where all the quantities that are labelled with primes are associated to the renormalized lattice, which is isomorphic to the original one scaled by a factor b. Furthermore the origin of energies is also renormalized, i.e. a new term added to $\mathcal{H}'(S')$ appears; we shall refer to it as the G-constant though it depends in the fact on the the initial dimensionless coupling constants. Furthermore the partition functions satisfy:

$$\sum_{\{S\}} \exp[\mathcal{H}(S)] = \sum_{\{S'\}} \exp[G + \mathcal{H}'(S')] ,$$

where the sum runs over all spin configurations. If we note F and F' the dimensionless free energies of the system, before and after renormalization, we obtain:

$$F = G + F' . \qquad (1)$$

Let us note N and N' the number lattice sites respectively before and after renormalization ($N/N' = b^d$, d being the dimensionality of the lattice). Then in the thermodynamic limit we have that $F = Nf(K,H,...)$ and that $F' = N'f(K',H',...)$, where f is the free energy per site. By introducing the definition $G = Ng$, we can rewrite relation (1) as follows:

$$f(K,H,...) = g(K,H,...) + b^{-d} f(K',H',...) . \qquad (1')$$

This basic relation will be used later to obtain the internal energy, the specific heat and the spontaneous magnetization as functions of temperature. Let us also remark that up to this point no particular choice has been assumed for the RGT, which can be obtained by cumulant expansions, decimation on the whole lattice or decimation on finite clusters (which can be either of the standard type, i.e. subsets of the real lattice, or of the type used herein, i.e. cells which somehow simulate basic symmetries of the real lattice).

2.2 Self-dual cluster renormalization group approach

a) Internal energy and specific heat

For the nearest-neighbour $\frac{1}{2}$ - spin Ising model we have (in the absence of any magnetic field):

$$\mathcal{H}(S) = K \sum_{<i,j>} S_i S_j \quad , \qquad (2)$$

If we exclude the appearance of new coupling constants, the renormalized hamiltonian will be given by

$$\mathcal{H}'(S') = K' \sum_{<i',j'>} S_{i'} S_{j'} \quad . \qquad (3)$$

Let us now use the decimation method for self-dual clusters to obtain, for the square lattice, K'(K) (see for example Yeomans and Stinchcombe 1979, 1980, Levy et al 1980, Tsallis and Levy 1980, Curado et al 1981) and $K'_o(K)$. For example, for $b = 2$, we obtain the following relation:

$$\exp\left(K'_o + K'\mu_A\mu_B\right) = \sum_{\{\sigma_1,\sigma_2\}} \exp\left[K\left(\mu_A\sigma_1 + \mu_A\sigma_2 + \sigma_1\sigma_2 + \mu_B\sigma_1 + \mu_B\sigma_2\right)\right] , \tag{4}$$

where the μ_i and σ_i spins are represented in Fig. 1, and the sum runs over the $\{\sigma_i\}$ configurations. Therefore

$$K' = \frac{1}{2} \ln (W/R) ,$$

$$K'_o = \frac{1}{2} \ln (W\ R) ,$$

where

$$W \equiv \exp(5K) + 2\exp(-K) + \exp(-3K) ,$$

$$R \equiv 4 \cosh K.$$

In other words, by decimating over spins 1 and 2 of the self-dual cluster of Fig. 1(a), the renormalized self-dual cluster of Fig. 1(b) is obtained. This transformation preserves duality, therefore, the exact critical point is recovered: $K'(K)$ admits as an unstable fixed point $K_c = K_c^{\text{exact}} = \frac{1}{2} \ln(\sqrt{2} + 1)$. The other fixed points (stable) are $K = 0$ and $K = \infty$. These three fixed points remains invariable for all b.

In order to close the procedure (which will enable us to calculate the internal energy and the specific heat) we need to relate the G-constant to K'_o. Because of extensivity and since both K'_o and G represent that part which does not depend on spin variables we propose the following relation among them:

$$G = D\, N\, K'_o ,$$

hence

$$g = D\, K'_o ,$$

where D is a constant which is related to the fact that the self-dual cluster sites do not coincide with the lattice sites and has to be fixed. Therefore our present approximation of Eq. (1') is

$$f(K) = D\, K_o'(K) + b^{-d} f(K') . \tag{5}$$

Through successive derivation we obtain:

$$\frac{df}{dK} = D \frac{dK_o'}{dK} + b^{-d} \frac{df}{dK'} \frac{dK'}{dK} , \tag{6}$$

and

$$\frac{d^2f}{dK^2} = D \frac{d^2K_o'}{dK^2} + b^{-d} \left[\frac{d^2f}{dK'^2} \left(\frac{dK'}{dK}\right)^2 + \frac{df}{dK'} \frac{d^2K'}{dK^2}\right] . \tag{7}$$

The recursion relations (5), (6) and (7) enable us to numerically obtain, for both para- and ferro-magnetic phases, the free energy per site, the internal energy per site $U = - J(df/dK)$ and the specific heat per site $C = k_B K^2(d^2f/dK^2)$, once the constant D has been determined. It is impostant to remark that all three Eqs. (5), (6) and (7) are invariant through the transformation $D \to \lambda D$, $f \to \lambda f$, $K \to K$ (hence $df/dK \to \lambda df/dK$ and $d^2f/dK^2 \to \lambda d^2f/dk^2$), i.e. the constant D is nothing but a K-independent expansion factor of the quantities f, df/dK and d^2f/dK^2 (therefore the values of the critical exponents do not depend on D). There are many manners to determine D (all of them are expected to be equivalent for $b \to \infty$): we restrict our analysis to three of them. Let us first consider the fixed point $K = \infty$: it is straightforward to obtain

$$\left.\frac{dK'}{dK}\right|_{K=\infty} = b ,$$

(particular case of the relation $(dK'/dK)_{K=\infty} = b^{d-1}$ discussed by Klein et al 1976) and

$$\left.\frac{dK'_o}{dK}\right|_{K=\infty} = (b-1)(2b-1),$$

hence (through use of Eq. (6))

$$D = \frac{(df/dK)_{K=\infty}}{b(2b-1)} = \frac{2}{b(2b-1)}, \tag{8}$$

where we have used the exact value of $(df/dK)_{K=\infty}$ (easy to obtain). We can also notice that Eq. (7) is identically satisfied due to the fact that both d^2K'/dK^2 and $d^2K'_o/dK^2$ vanish in the limit $K=\infty$ (as a matter of fact d^2f/dK^2 vanishes as well in this limit because df/dK tends to a constant).

Let us now consider the fixed point $K=0$: it is straightforward to obtain that

$$K'_o(0) = b(b-1)\ \ln 2\ ,$$

hence (through use of Eq. (5))

$$D' = \frac{b+1}{b^3\ \ln 2}\ f(0) = \frac{b+1}{b^3}\ , \tag{9}$$

where we have used the exact value of $f(0)$ (easy to obtain) and have introduced the notation D' in order to avoid confusion with the previous determination of the constant D. We can also remark that dK'/dK and dK'_o/dK vanish for $K=0$, therefore (through Eq. (6)) df/dK vanish as well, which is the exact result. Let us now describe the third determination (noted D'') of the constant D. If we use, into Eq. (7), the facts that df/dK and dK'/dK vanish and d^2K'/dK^2 is finite for $K=0$ together with

$$\left.\frac{d^2K'_o}{dK^2}\right|_{K=0} = 2b^2 - 2b + 1\ ,$$

we obtain

$$D'' = \frac{(d^2f/dK^2)_{K=0}}{2b^2 - 2b + 1} = \frac{2}{2b^2 - 2b + 1} , \tag{10}$$

where we have used the exact result of $(d^2f/dK^2)_{K=0}$ (easy to obtain).

The ratio D'/D and D''/D (obtained from Eqs. (8), (9) and (10)) tend to unity for $b \to \infty$ as expected; furthermore they monotonically decrease for b increasing (their values for $b = 2$ respectively are 9/8 and 6/5). It is therefore clear that the use of one or another determination of D introduces nothing but small numerical differences in the thermal behaviours we are interested in, differences which vanish in the limit $b \to \infty$. The results presented in this work (for $b = 2, 3, 4$ and 5; see section 3) have been obtained through use D given by Eq. (8). Let us add also that the fixed point $K = K_c$ could have been used to fix D by asking

$$f(K_c) = f^{\text{exact}}(K_c) ,$$

or

$$(df/dK)_{K_c} = (df^{\text{exact}}/dK)_{K_c} .$$

However this strategy demands the knowledge of the exact solution of the problem (in this particular case the Onsager (1944) result) and therefore greatly restricts its applications.

b) Spontaneous magnetization

Let us now add the term $H \sum_i S_i$ to hamiltonian (2), and consequently the term $H' \sum_{i'} S_{i'}$ to hamiltonian (3). Equation (4) generalizes into

$$\exp\left[K'_0 + K'\mu_A\mu_B + H'(\mu_A + \mu_B)\right]$$

$$= \sum_{\{\sigma_i\}} \exp\left[K\,\Pi_{\mu_A\mu_B} + H\,p_A(\mu_A+\mu_B) + H\sum_{i=1}^{b(b-1)} p_i\sigma_i\right] , \tag{11}$$

with

$$\Pi_{\mu_A\mu_B} \equiv \begin{cases} \mu_A(\sigma_1 + \sigma_2) + \mu_B(\sigma_1 + \sigma_2) + \sigma_1\sigma_2 & \text{if } b=2 \\ \mu_A \sum_{i=1}^{b} \sigma_i + \mu_B \sum_{i=b+1}^{2b} \sigma_i + \sum_{<i,j>} \sigma_i\sigma_j & \text{if } b > 3 \end{cases},$$

where the A, B and i-th (for $i = 1,2,\ldots, b(b-1)$ sites are illustrated for $b = 5$ in Fig. 2, and p_A and $\{p_i\}$ are topological weights. The RGT transforms b^2 sites of the original lattice into one site of the renormalized lattice. However at the cluster level weights must be introduced. In what concerns each one of the terminal sites (A and B), b different original sites have been collapsed into one therefore

$$p_A = b.$$

Furthermore if we consider the whole cluster, the renormalization proportion b^2 into 1 must be preserved, and as the renormalized cluster has only 2 sites, it must be

$$\sum_{i=1}^{b(b-1)} p_i + 2p_A = 2\, b^2. \qquad (12)$$

Let us stress that topologically equivalent sites have the same weight p_i (for example sites 1,5,6 and 10 in Fig. 2).

If we consider the configuration $\mu_A=\mu_B=1$ in Eq. (11) and differentiate with respect to H, we obtain

$$\left(\frac{\partial H'}{\partial H}\right)_{H=0} = p_A + \sum_{i=1}^{b(b-1)} \frac{p_i}{2} <\sigma_i> , \qquad (13)$$

with

$$<\sigma_i> \equiv \frac{\sum_{\{\sigma_j\}} \sigma_i \exp\left[K\, \Pi_{11}\right]}{\sum_{\{\sigma_j\}} \exp\left[K\, \Pi_{11}\right]} , \qquad (13')$$

where we have used the fact that K'_o and K' are even functions of H. Furthermore Eq.(5) generalizes into

$$f(K,H) = D\,K'_o\,(K,H) + b^{-d}\,f(K',H') \quad ,$$

and since the dimensionless spontaneous magnetization is given by $m(K) = (df/\partial H)_{H=0}$, we obtain the following recursive relation

$$m(K) = b^{-d}\,m(K')\left(\frac{\partial H'}{\partial H}\right)_{H=0} \quad , \tag{14}$$

which, together with Eq.(13) (where a particular choice of $\{p_i\}$ has to be done), enable us for numerical calculation of the thermal dependence of the spontaneous magnetization. At the fixed point $K=\infty$, $<\sigma_i>$ equals unity, therefore the use of relation (12) into (13) leads to

$$\left.\frac{\partial H'}{\partial H}\right|_{H=0} = b^2 \; . \tag{15}$$

This equation transforms Eq.(14) into an identity, as it should be in order to allow a finite non vanishing value for the spontaneous magnetization. At the fixed point $K=0$, $<\sigma_i>$ vanishes , therefore, through Eq.(13), $(\partial H'/\partial H)_{H=0}$ equals b *for all choices of the weights* $\{p_i\}$.

Let us now discuss the possible choices for the weight $\{p_i\}$ (we recall that $p_A=b$). The simplest possibility (referred to as criterion (a)) clearly is

$$p_i = p = 2 \qquad \forall i \;,$$

where the sum rule (12) has been used. The next simplest possibility (referred to as criterion (b) and different from the preceding one only for $b \geqslant 3$), namely to introduce two different weights (noted q and r), enables us for partial consideration of the topological differences between the cluster sites. We shall assume

$p_i = q$ if the coordination number of the i-th site is equals 3,

and

$p_i = r$ if the coordination number of the i-th site is equals 4,

with the restriction

$q/r = 3/4$ (ratio of the coordination numbers).

Within this assumption Eq.(12) leads to

$$r = \frac{4b}{2b-1} \qquad (b \geqslant 3) \ . \qquad (16)$$

It is important to remark that, for a given cluster, there are 2(b-1) sites whose weight is q and (b-1)(b-2) sites whose weight is r. Therefore, in the limit $b \to \infty$, the latter dominate, and as $r \to 2$ (see Eq.(16)), *both criteria become equivalent for all values of* K. The results associated to both criteria are pre sented in the next section; nevertheless it is worth while to anticipate here that criterion (b) is more performant.

3. Results

a) Internal energy and specific heat

We have made calculations for clusters with b = 2, 3,4 and 5 by using D given by Eq.(8). The computation of the internal energy, the specific heat and the spontaneous magnetization for the cluster with b = 5 involves evaluating sums as in Eqs. (11) and (13') over 2^{20} states. We have used a program which considerably shortens the computing time (Vucetich 1980).

In Fig. 3 the percentage error of the internal energy (defined as $E \equiv 100\left[(df/dK) - (df/dK)^{exact}\right]/(df/dK)^{exact}$) is shown for $b = 2,5$. This error vanishes in the limit $K \to \infty$. In the opposite limit ($K = 0$) it is given by

$$E = 50 \left[\left.\frac{d^2 f}{dK^2}\right|_{K=0} - 2 \right] = - \frac{100(b-1)}{b(2b-1)}$$

which monotonically vanishes for $b \to \infty$ (this is clearly related to the fact that the ratio D''/D tends to unity). The behaviour (with b increasing) of the internal energy at the critical point is indicated in table 1.

In Fig. 4 the specific heat is shown. As we have already mentioned, the exact K_c value is obtained. In the K-dominion corresponding to Fig. 4 ($0 < K \lesssim 0.9$), the results improve when the cluster size increases. This improvement in K_c is shown in table 1 (where the critical exponents y_T and α are indicated as well). Far from K_c, for $K \gtrsim 1.0$, there appears a zone where the approximate specific heat takes slightly negative (unphysical) values. This is shown in Fig. 5(a) (notice the scale amplification). When we pass from $b = 2$ to $b = 3,4$ the curve gets worse. For $b = 5$ the curve begins to improve; this could indicate that for large clusters the specific heat negative values would disappear.

To estimate what happens in the limit $b \to \infty$, we have made an extrapolation for each value of K with a polynomial of third degree in $1/b$ using the results obtained for clusters with $b = 2,3,4$ and 5. In order to analyze the tendency of the extrapolation, we have also used two polynomials of the second degree. One of them is adjusted for $b = 2,3$ and 4; the other one is adjusted for $b = 3,4$ and 5. These results are shown in figure 5(b). We can see that even the case with $b = 2,3$, and 4, improves compared with the cluster with b=5 (the width of the negative curve decreases). In general the extrapolation tendencies indicate that the unphysical negative specific heat will disappear when $b \to \infty$ (the difficulties encountered here might be related to the similar ones encountered by Dunfield and Noolandi 1980).

b) Spontaneous magnetization

We have calculated the spontaneous magnetization for clusters with $b = 2,3,4$ and 5, within both criteria (a) and (b). In all cases, if we start the iterative procedure (Eq.(14)) with $m(K) \neq 0$ for $K < K_c$, we obtain that the magnetization diverges in $K = 0$ $(T = \infty)$, therefore it must be $m(K) = 0$. For $K > K_c$, in view of Eq.(15), we obtain a finite spontaneous magnetization.

.We verify that for both criteria, the spontaneous magnetization curve improves when the cluster size increases. For a given cluster size, the curve obtained with criterion (b) is better than that obtained with criterion (a). In Fig. 6 we present the worst curve, which corresponds to $b = 2$, where both criteria are one and the same (see Fig. 1(a)), as well as the best curve obtained, which corresponds to $b = 5$ and criterion (b). We remark that the difference between these curves is quite small.

In table 2, we present the critical exponents y_H and β for different clusters, using criteria (a) and (b). We remark that criterion (a) leads to y_H values which get worse when b increases, and to β values which get better, whereas criterion (b) leads to a good behaviour for both exponents (this is not surprising if we remember that the topological differences between cluster sites are better taken into account within criterion (b) than (a)). One can speculate that the bad behaviour

The percentage errors of the worst ($b = 5$ and criterion (a)) and best ($b = 5$ and criterion (b)) values for y_H are respectively -0.41% and -0.15% .One can speculate that the bad behaviour of y_H within criterion (a) will reverse for sufficiently large value of b, and that its value will improve (as it happens with the negative specific heat, see Fig. 5(a)). To estimate the results that we should obtain using large clusters, we have extrapolated y_H (see table 3) in the same way we did for negative specific heat. We can see from I, II and III extrapolated values that the difference between both criteria gets smaller (as expected from the consequences of Eq. (16)).

4. Summary and conclusions

Until now (Yeomans and Stinchcombe 1979, 1980, Tsallis and Levy 1980, Curado et al 1981) the RGT based on self-dual cluster have been used to obtain the renormalization nearest-neighbour coupling K'(K) in the square lattice Ising model. The exact critical point is obtained and the number of coupling constant does not increase through the transformation. Within the framework of the present clusters, some constants (D, p_A and $\{p_i\}$ have to be determined in order to calculate the internal energy U, specific heat C and magnetization. The constant D is determined by imposing the correct departure (of U or C for example) either on $K=0$ or $K=\infty$ (quantities easy to calculate, and which by no means demand the knowledge of exact solutions as the Onsager one in the present case). In what concerns the constants p_A and $\{p_i\}$ they are determined through simple topological considerations (essentially that at the cluster level, the renormalization proportion b^2 lattice sites into one lattice site must be preserved).

For $b=2$, the present calculation is of great simplicity, due to the fact that sums of the type Eqs. (4) and (13') have only 2^2 terms. The results are quite satisfatory, in particular close to the critical point, where they are in excellent agreement with the exact values. Let us now compare our results for $b=2$ with those obtained through other approximations. A simple variational approach to the eigenvalue problem of the transfer operator has been proposed by Rujan (1979). This author obtains the specific heat, the spontaneous magnetization (both, when K is not very close to K_c^{exact}) and the internal energy curves with high accuracy, but the critical point is not exact ($K_c = 0.413$, 0.422 in his first and second approximations, respectively; $K_c^{exact} \simeq 0.4407$), and β is about 0.4 (for both approximations). In our case, β is 0.1486 ($\beta^{exact} = 0.125$), also here the internal energy is very accurate. On the other hand, the Rujan results of specific heat for $K \gtrsim 0.5$ and spontaneous magnetization for $T/T_c^{exact} \lesssim 0.95$ are better than ours. Jayaprakash et al (1978) have used the Migdal-Kadanoff approximate recursion relation for randomly bond-dilute in square Ising model. Although their main interest is to analyze the influence of dilution, their particular case $p = 1.0$ (see Fig.7 of their paper) can be compared with the present results. They obtain the exact critical point, the inverse dimensionless specific heat $\left[K^2 \, (d^2f/dK^2)\right]_{K_c}^{-1} \simeq 1$ and the critical exponent $\alpha = -0.654$. We obtain, $\left[K^2 (d^2f/dK^2)\right]_{K_c}^{-1} = 0.4895$ and $\alpha = -0.2973$ (the exact values are 0 and 0(ln) respectively).

Let us now discuss the dependence of our results on cluster size. The curve of specific heat near K_c ($0 < K \lesssim 0.9$), see Fig. 4 and table 1) and the curve of spontaneous magnetization (using both criteria) for all K, improve when the size of the cluster increases from $b = 2$ to $b = 5$. For values far from K_c ($K \gtrsim 1.0$) we obtain a slightly negative specific heat. Extrapolation considerations suggest that this defect would tend to disappear for $b \to \infty$. In what concerns the critical exponents, y_H through criterion (b), y_T, α and β (through both criteria) present the correct tendency for b growing from $b = 2$ to $b = 5$, whereas y_H through criterion (a) (which is rather rough from the topological stand-point) presents the wrong tendency at least until $b = 5$ (as both criteria become equivalent for $b \to \infty$, one can speculate that this tendency will be reversed for larger clusters, as suggested by simple extrapolation considerations). The convergence of real space renormalization groups is a rather delicate point : see Griffiths and Pearce (1979) for general remarks and Sneddon and Barber (1977) for the particular case of decimation (although the decimation discussed therein is quite different from the present one, and therefore it is not obvious that their conclusions could straightforwardly apply to the present case. However an increasing amount of concrete treatments (Ising as well as percolation model: Reynolds et al 1979, Curado et al 1981, Eschbach et al 1980 among others) suggest that convergence towards the exact solutions does exist, at least for the present type of real space renormalization group.

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Figure Caption

Figure 1 Renormalization of the cluster with $b = 2$. The renormalized nearest-neighbour iteration $K'(K)$ is obtained by renormalizing the self-dual cluster (a) to the (b) one. The μ_A, μ_B and σ_i $(i = 1,2)$ are the site spins of clusters.

Figure 2 Renormalization of the cluster with $b = 5$ when the magnetic interaction is present. The assign ment of the magnetic field and the spin related to each i-th site, $i = 1, \ldots 20$, are $p_i H$ and σ_i respectively.

Figure 3 The percentage errors of the internal energy;

$$E = \left[(df/dK) - (df^{exact} / dk\right] 100 / (df^{exact}/dK).$$

Doted line: the $b = 2$ case, full line: the $b = 5$ case.

Figure 4 The reduced specific heat K^2 (d^2f/dK^2). Doted line: exact value (Onsager 1944); full line: our result; (a) with $b = 2$ and (b) with $b = 5$.

Figure 5 The negative $K^2(d^2f/dK^2)$ values obtained (notice the scale amplification with respect to Fig. 4). Dashed-dot-dot line: exact value (Onsager 1944). (a) our results obtained with different cluster sizes, dashed-dot line: with $b = 2$, doted line: with $b = 3$, dashed line: with $b = 4$ and full line: with $b = 5$. (b) extrapolation to the cluster with $b \to \infty$. Doted line: using the results obtained from clusters with $b = 2,3$ and 4; dashed line: the same with $b = 3,4$ and 5; full line: the same with $b = 2,3,4$ and 5.

Figure 6 The spontaneous magnetization. Doted line: the $b = 2$ case, where the two criteria are the same (see Fig. 1(a)); full line: the $b = 5$ and criterion (b) case; dashed line: exact value (Yang 1952).

Table Caption

Table 1 Values related to the internal energy ($U = -J(df/dK)$) and the specific heat ($C = k_B K^2 (d^2f/dK^2)$), the thermal y_T ($(dK'/dK) = b^{y_T}$) and α ($\alpha = 2 - d/y_T$) exponents obtained with different cluster sizes.

Table 2 The magnetic y_H ($(\partial H'/\partial H)_{\substack{H=0 \\ H=K_c}} = b^{y_H}$) and β ($\beta = (d - y_H)/y_T$) exponents.

Table 3 Extrapolation of y_H for $b \to \infty$. Value I: obtained by making an extrapolation with the results from clusters with $b = 2,3$ and 4; value II: the same with $b = 3,4$ and 5; value III: the same with $b = 2,3,4$ and 5. The values I and II are calculated in order to analyze the tendency of the extrapolation.

Table 1

b	$\left(\frac{df}{dK}\right)_{K_c}$	$\left[K^2 \frac{d^2f}{dK^2}\right]^{-1}_{K_c}$	y_T	α
2	1.3770	0.4895	0.8706	-0.2973
3	1.3866	0.3786	0.9014	-0.2187
4	1.3922	0.3377	0.9132	-0.1900
5	1.3956	0.3173	0.9192	-0.1758
Exact	$\sqrt{2} \simeq 1.4142$	0	1	0(ln)

Table 2

b	Criterion (a)		Criterion (b)	
	y_H	β	y_H	β
2	1.8706	0.1486	1.8706	0.1486
3	1.8681	0.1463	1.8715	0.1426
4	1.8674	0.1452	1.8720	0.1402
5	1.8673	0.1444	1.8722	0.1390
Exact	1.875	0.125	1.875	0.125

Table 3

Extrapolation	y_H	
	Criterion (a)	Criterion (b)
I	1.8676	1.8739
II	1.8690	1.8724
III	1.8700	1.8714
Exact	1.875	

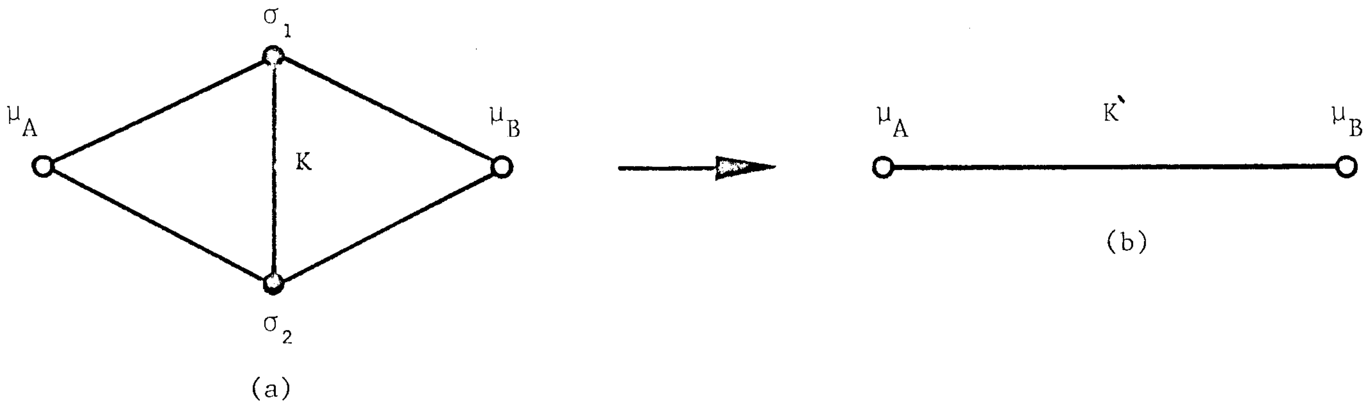

Figure 1

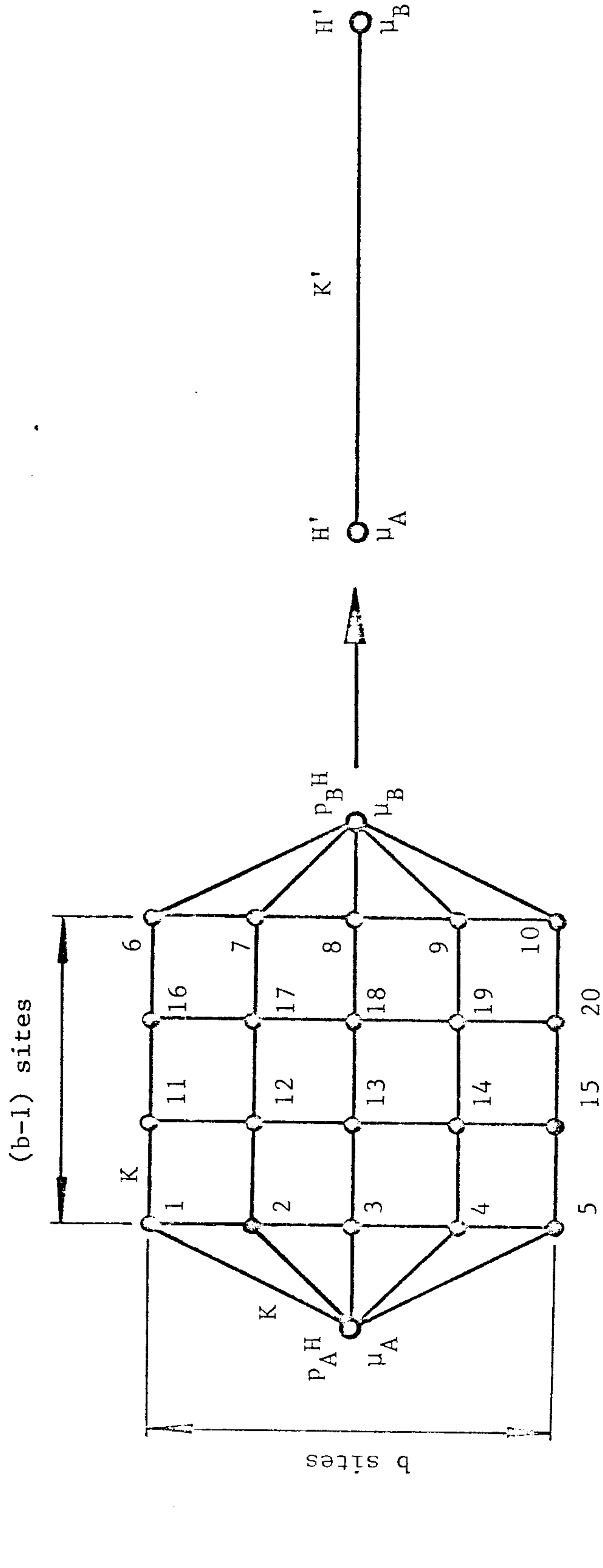
(b-1) sites
b sites
K
K
$p_A H$
μ_A
$p_B H$
μ_B
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
H'
μ_A
K'
H'
μ_B

Figure 2

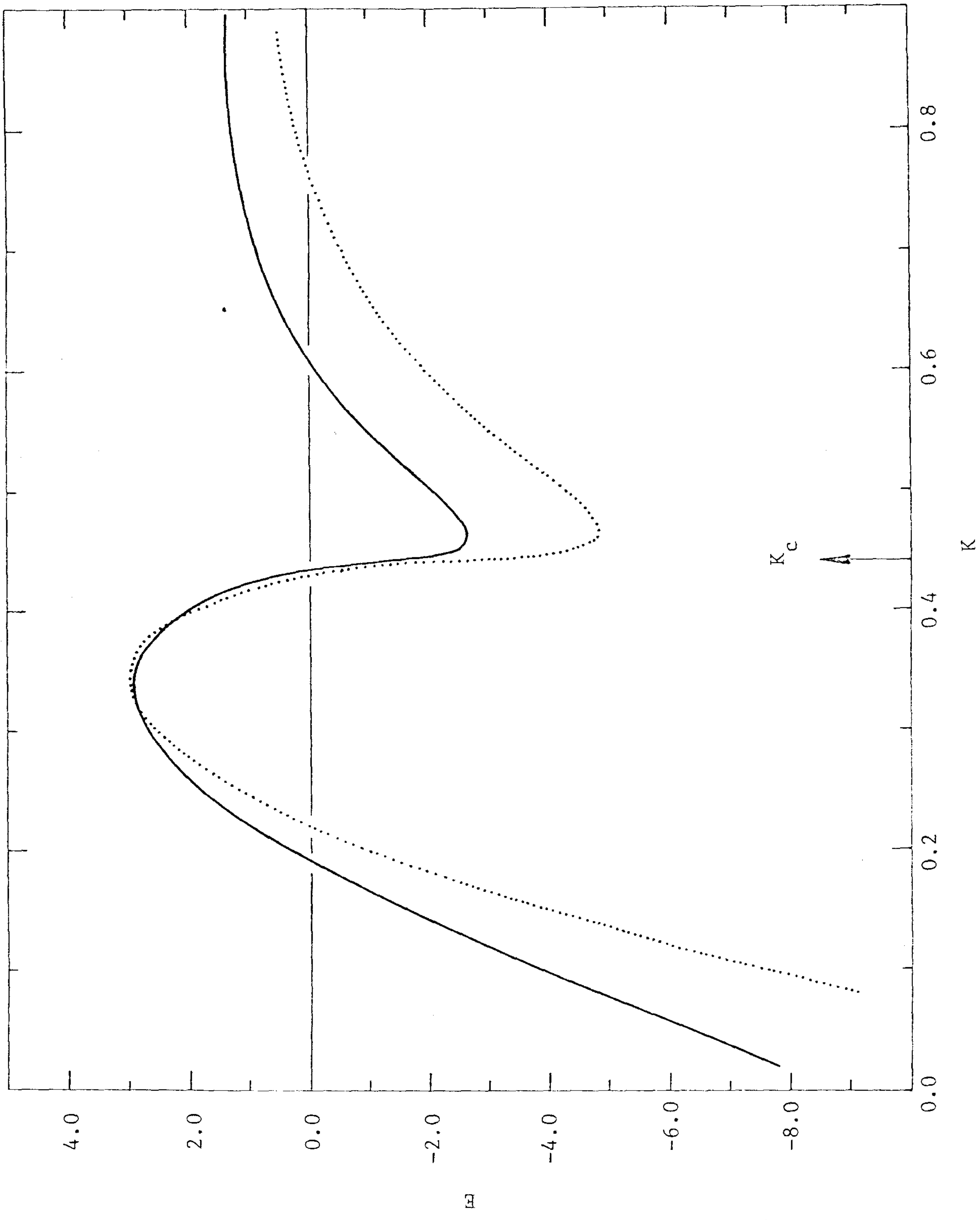

Figure 3

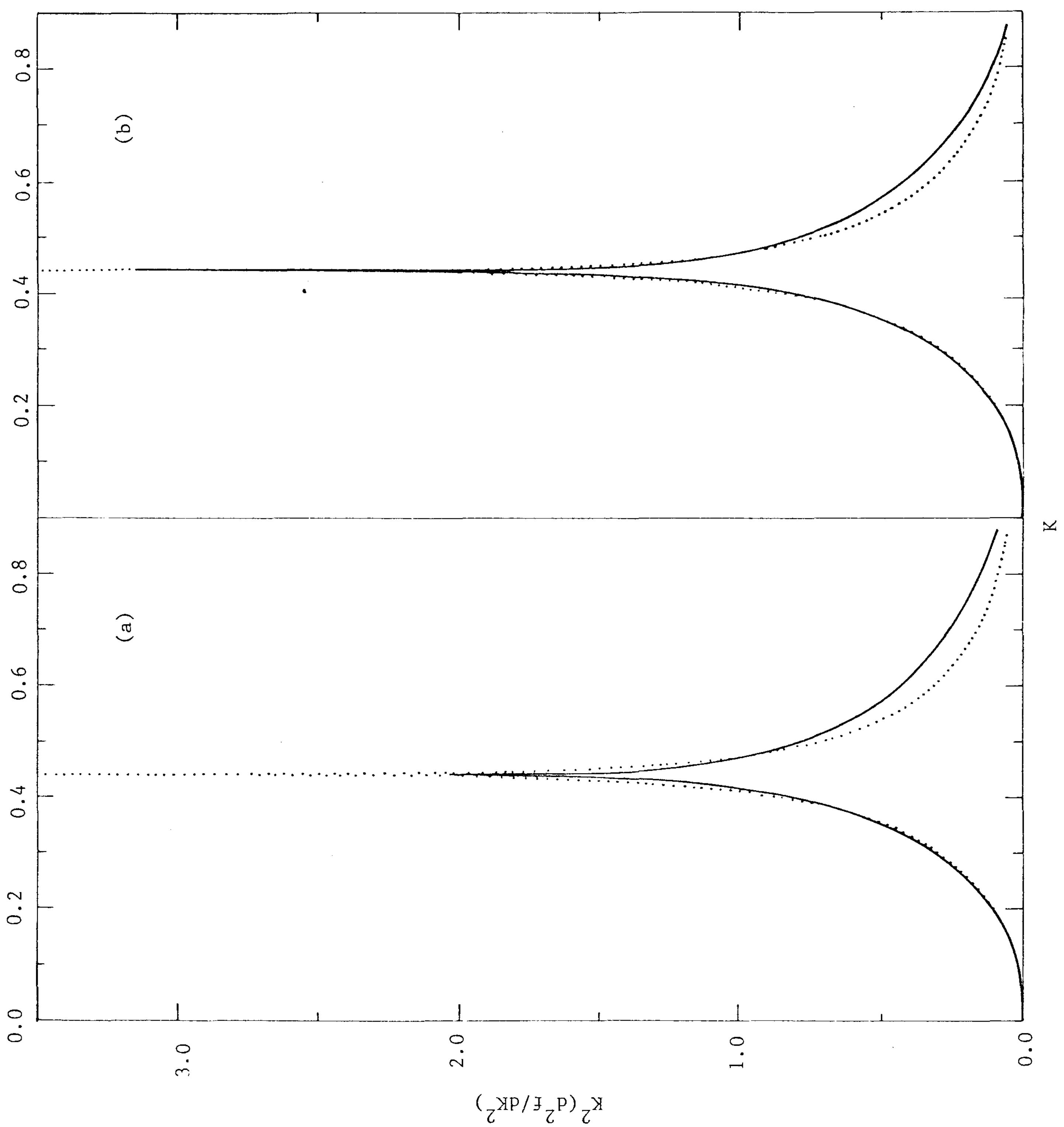

Figure 4

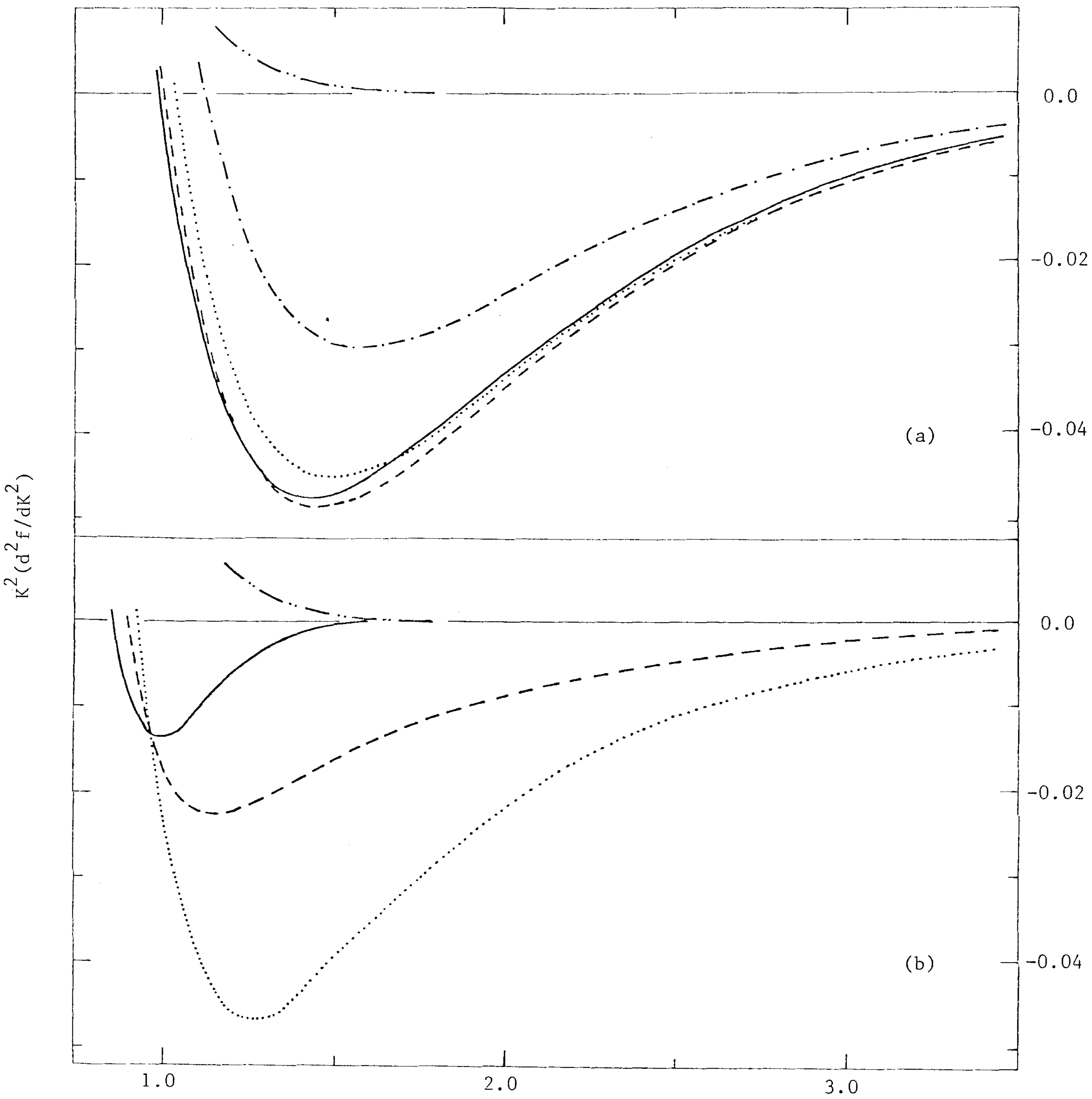

Figure 5

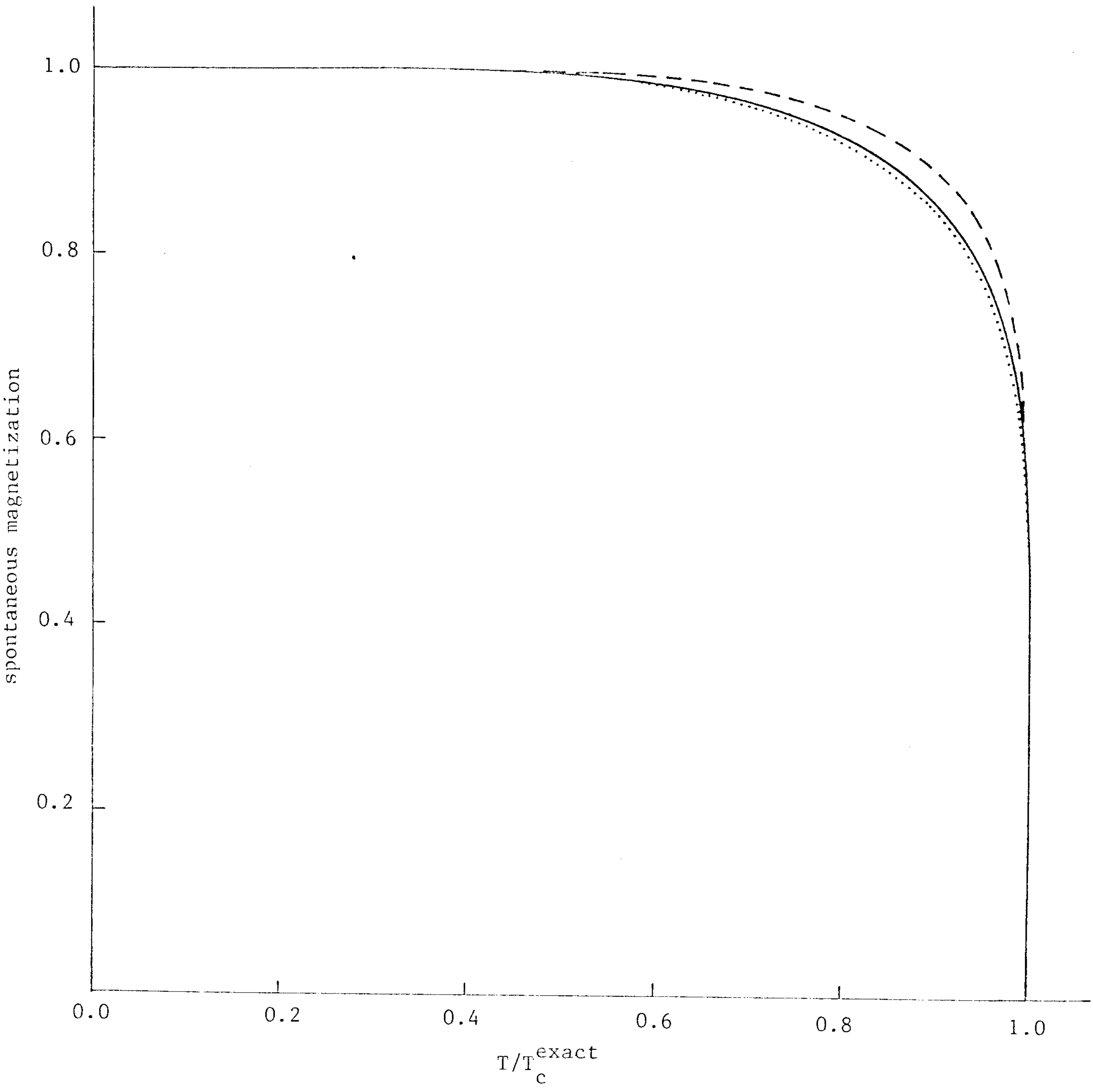
1.0
0.8
0.6
0.4
0.2
spontaneous magnetization
0.0
0.2
0.4
0.6
0.8
1.0
T/T_c^{exact}

Figure 6