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EXPRESSIONS FOR THE CURRENT OF THE DIRAC THEORY

by

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ABSTRACT - Expressions for the current j_μ of the Dirac theory are obtained from the invariance of the Lagrangian of the theory under given transformations. It is shown that j_μ is a vector with respect to Lorentz transformations.

The problem of obtaining new invariance algebras for relativistic equations, by using a non-Lie approach, has been considered by several authors[1-6]. The application made in reference (6) to the Dirac equation has some consequences worth of discussion. It is the aim of the present note to show that the Lagrangian of the Dirac theory can be shown to be invariant under given transformations. As a consequence a conservation law holds so that expressions for the corrent j_μ can be obtained. These expressions are different from the ones analyzed by Laporte, Uhlenbeck and others[7]. Moreover, it is shown that j_μ behave as the components of a vector under Lorentz thansformations.

The Lagrangian L of the Dirac theory is

(1) $$L = i\,\overline{\Psi}(\gamma_\mu\,\partial_\mu + m)\Psi$$

We shall proof that this Lagrangian is invariant under the transformation

(2) $$\Psi \rightarrow \Psi' = \exp\left[i\,A_{j\ell r}\,\Theta_{j\ell r}\right]\Psi ,$$

(without sum!). $\Theta_{j\ell r}$ are parameters and the quantities $A_{j\ell r}$ are defined by

(3) $$A_{j\ell r} = V^{-1}_{j\ell}\,A'_{jr}\,V_{j\ell}.$$

A'_{jr} are Hermitean operators of square one which commute with γ_4,

(4) $$\left[A'_{jr}, \gamma_4\right] = 0 .$$

$V_{j\ell}$ are unitary matrices

(5) $$V^{+}_{j\ell}\, V_{j\ell} = V_{j\ell}\, V^{+}_{j\ell} = 1,$$

which satisfy the relation

(6) $$\left[V_{j\ell},\, \gamma_4\right] = 0$$

We shall use the following representation

$$\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \gamma_K = \begin{pmatrix} 0 & \sigma_K \\ \sigma_K & 0 \end{pmatrix} .$$

The matrix $Q = -i\gamma_4\, \gamma_K\, p_K$ is transformed by means of $V_{j\ell}$ into the matrices given in table (II). The transformation is

(7) $$i V_{j\ell}\, Q\, V^{-1}_{j\ell} = Q'_j$$

For each Q'_j there are several transformations V_j and for each Q'_j there are four operators $A'_{j\ell}$. We are indicating in the table (II) some matrices $V_{j\ell}$ that satisfy (6) and (7). In Q there appears the scalar product

$$(\sigma,p) = \sigma_1\, p_1 + \sigma_2\, p_2 + \sigma_3\, p_3,$$

and in $V_{j\ell}$ there appears also the quantity

$$p = (p_1^2 + p_2^2 + p_3^2)^{1/2}.$$

The proof of the invariance of $\mathcal{L}$ under the transformation (2) is easy to be obtained. We have in the case of the transformation

(8) $$\Psi \rightarrow \Psi'' = A_{j\ell r}\,\Psi,$$

$$\mathcal{L} \rightarrow \mathcal{L}'' = i\,\overline{\Psi}''\,(\gamma_\mu \partial_\mu + m\,)\,\Psi''$$

(9) $$= i\,\Psi^{HC}\,\gamma_4\,A_{j\ell r}(\gamma_\mu \partial_\mu + m)\,A_{j\ell r}\,\Psi,$$

in view of (4), (5), (6) and the assumed Hermiticity of A'_{jr} . According to the definition (3) of $A_{j\ell r}$, the relation (7), as well as the assumption that

(10) $$(A'_{j\ell})^2 = 1$$

is a relation that holds for all $A'_{j\ell}$, we can write (9) as

(11) $$\mathcal{L} \rightarrow \mathcal{L}'' = i\,\overline{\Psi}(\gamma_\mu \partial_\mu + m)\Psi = \mathcal{L}$$

Therefore, the Lagrangian is invariant under (8). To show that the general relation (2) is also true it is enough to take (11) into account, and write (2) as

(12) $$\Psi' = (\cos\Theta_{j\ell r} + i\,A_{j\ell r}\,\mathrm{sen}\Theta_{j\ell r})\Psi$$

which is possible in view of (10).

As a consequence, if $\Theta_{j\ell r}$ is considered to be infinitesimal, the following law can be shown to hold:

(13) $$\partial_\mu(\overline{\Psi}\,\gamma_\mu\,A\,\Psi) = 0$$

This relation can be proved also by applying $A_{j\ell r}$ to the left-hand side of the Dirac equation. When we manipulate the equation as shown before we obtain a wave function $A_{j\ell r}\Psi$. The adjoint equation can be obtained and the relation (13) is easily deduced.

If Q_j is given there are four possible operators $A'_{j\ell}$. One of them is $A_{j\ell} = I$ so that $A_{j\ell r} = I$ and the ordinary con - servation law is obtained. When, however, we consider the remaining three $A'_{j\ell}$ we obtain an expression for j_μ that is different from the usual one. It is easy to see that the quantities on the same line of table (II) satisfy

(14) $$\left[A'_{jr} , A'_{js}\right] = 2i\, \varepsilon_{rst}\, A'_{jt} .$$

We shall proof now that the quantities j_μ transform, under Lorentz transformations, as components of a vector. For simplicity we shall neglect, from now on, the indices of $A_{j\ell r}$. It is well known that, under the transformation

(15) $$x'_\mu = a_{\mu\nu}\, x_\nu ,$$

the equation

(16) $$(\gamma_\mu\, \partial'_\mu + m)\, \Psi' = 0 ,$$

is transformed into

(17) $$(\gamma_\mu\, \partial_\mu + m)\, \Psi = 0 .$$

Suppose now that (16) is multiplied from the left by a quantity A' that commutes with $\gamma_\mu \partial'_\mu$. It is easy to see that we have

$$(18) \qquad (\gamma_\mu \partial'_\mu + m) A' \Psi' = 0 ,$$

and

$$(19) \qquad (\gamma_\mu \partial_\mu + m) A \Psi = 0 ,$$

instead of (16) and (17), where

$$(20) \qquad A = S^{-1} A' S,$$

$$(21) \qquad \Psi' = S \Psi ,$$

$$(22) \qquad a_{\mu\nu} S^{+1} \gamma_\nu S^{-1} = \gamma_\mu$$

It is easy to see that the quantity A given in (20) commutes with $\gamma_\mu \partial_\mu$. In order to proof this statement it is enough to go back from (19) to (18), use the assumed commutation relation

$$[\gamma_\mu \partial_\mu , A'] = 0$$

and perform a Lorentz transformation.

Finally, the proof that j_μ behaves as a vector can be made as follows. We have

$$j'_\mu = \overline{\Psi}' \gamma_\mu A' \Psi' ,$$

and

$$j_\mu = \overline{\Psi}\,\gamma_\mu\, A\, \Psi \quad ,$$

so that in view of (20), (21) and (22) we have

$$j'_\mu = a_{\mu\nu} \;\; j_\mu$$

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Q'_j	A'_{j1}	A'_{j2}	A'_{j3}	A'_{j4}
$\pm p\ \gamma_4\ \gamma_1$	$-i\ \gamma_2\ \gamma_3$	$-i\ \gamma_2\ \gamma_5$	$-i\ \gamma_3\ \gamma_5$	I
$\pm p\ \gamma_4\ \gamma_2$	$-i\ \gamma_1\ \gamma_3$	$-i\ \gamma_1\ \gamma_5$	$-i\ \gamma_3\ \gamma_5$	I
$\pm p\ \gamma_4\ \gamma_3$	$-i\ \gamma_1\ \gamma_2$	$-i\ \gamma_1\ \gamma_5$	$-i\ \gamma_2\ \gamma_5$	I
$\pm i\ p\ \gamma_1$	$-i\ \gamma_2\ \gamma_3$	$-i\ \gamma_2\ \gamma_5$	$-i\ \gamma_3\ \gamma_5$	I
$\pm i\ p\ \gamma_2$	$-i\ \gamma_1\ \gamma_3$	$-i\ \gamma_1\ \gamma_5$	$-i\ \gamma_3\ \gamma_5$	I
$\pm i\ p\ \gamma_3$	$-i\ \gamma_1\ \gamma_2$	$-i\ \gamma_1\ \gamma_5$	$-i\ \gamma_2\ \gamma_5$	I
$\pm p\ \gamma_4\ \gamma_5$	$-i\ \gamma_1\ \gamma_2$	$-i\ \gamma_1\ \gamma_3$	$-i\ \gamma_2\ \gamma_3$	I
$\pm i\ p\ \gamma_5$	$-i\ \gamma_1\ \gamma_2$	$-i\ \gamma_1\ \gamma_3$	$-i\ \gamma_2\ \gamma_3$	I

Table (II) - Matrices $A'_{j\ell}$

Q_j'	V_{j1}	V_{j2}
$\pm p\ \gamma_4\ \gamma_j$	$\begin{bmatrix} 1 & 0 \\ 0 & \mp\sigma_j(\sigma,p)/p \end{bmatrix}$	$\begin{bmatrix} \mp\sigma_j(\sigma,p)/p & 0 \\ 0 & 1 \end{bmatrix}$
$\mp i\ p\ \gamma_j$	$\begin{bmatrix} 1 & 0 \\ 0 & \mp i\ \sigma_j(\sigma,p)/p \end{bmatrix}$	$\begin{bmatrix} \pm i\ \sigma_j(\sigma,p) & 0 \\ 0 & 1 \end{bmatrix}$
$\pm p\ \gamma_4\ \gamma_5$	$\begin{bmatrix} 1 & 0 \\ 0 & \mp i(\sigma,p)/p \end{bmatrix}$	$\begin{bmatrix} \pm i(\sigma\ p)/p & 0 \\ 0 & 1 \end{bmatrix}$
$\pm ip\ \gamma_5$	$\begin{bmatrix} 1 & 0 \\ 0 & \mp(\sigma,p)/p \end{bmatrix}$	$\begin{bmatrix} \mp(\sigma,p)/p & 0 \\ 0 & 1 \end{bmatrix}$

Table (I) - Matrices V_j

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