ISSN 0029-3865

CBPF-NF-002/86

HOMOGENEOUS COSMOS OF WEYSSENHOFF FLUID IN EINSTEIN-CARTAN SPACE

by

J. Duarte de Oliveira, A.F.F. Teixeira and J. Tiomno

Centro Brasileiro de Pesquisas Físicas - CNPq/CBPF
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

HOMOGENEOUS COSMOS OF WEYSSENHOFF FLUID IN EINSTEIN-CARTAN SPACE

J. Duarte de Oliveira, A.F.F. Teixeira and J. Tiomno

Centro Brasileiro de Pesquisas Físicas
22290 - Rio de Janeiro, RJ - Brasil

ABSTRACT

The role played by the spin of matter in homogeneous Gödel-type universes is investigated. It is shown that the region of parametrization $m^2 \leq 4\Omega^2$ for solutions of Einstein's equations with fluid and fields (as obtained by Rebouças and Tiomno) can be extended to $m^2 > 4\Omega^2$, in Einstein-Cartan space. Properties of the solutions are examined, and comparison with related works is made.

PACS/ 86: 98.80.Dr., 04.40.+c, 04.50.+h, 04.20.Jb.

1. INTRODUCTION

In recent years an increasing interest is being focussed on space times of the Gödel-type.[1,2] These spaces have line element written in cylindrical coordinates as

$$ds^2 = (dt - H\,d\phi)^2 - dr^2 - D^2 d\phi^2 - dz^2 \quad , \tag{1}$$

where D and H are functions of r alone. For homogeneous spaces these functions must satisfy[2]

$$D''/D = m^2 = -\mu^2 = \text{const} \quad , \quad H'/D = 2\Omega = \text{const} \quad , \tag{2}$$

a prime meaning d/dr. Although the possibility exists for $-\infty < m^2 < +\infty$, solutions of Einstein's equations in presence of fluids and fields were obtained[2] only for $m^2 \leq 4\Omega^2$. The aim of this paper is to find physical sources for homogeneous Gödel-type spaces in the region $m^2 > 4\,\Omega^2$; preliminary and essential results are contained in Ref.[3].

Three nonequivalent classes of homogeneous solutions are possible for Eqs. (2),[2] according to whether D"/D is a positive (m^2), zero or a negative constant $(-\mu^2)$:

$$(+) : D = m^{-1}\sinh(mr) \quad , \quad H = (4\Omega/m^2)\sinh^2(mr/2) \quad , \tag{3}$$

$$(0) : D = r \quad , \quad H = \Omega r^2 \quad , \tag{4}$$

$$(-) : D = \mu^{-1}\sin(\mu r) \quad , \quad H = (4\Omega/\mu^2)\sin^2(\mu r/2) \quad . \tag{5}$$

It is customary to take the hyperbolic expressions (3) as standard expressions for the metric coefficients of homogeneous Gödel-type universes and say that the second class (algebraic) corresponds to m=0, while the third class (trigonometric) corresponds to negative m^2 (or imaginary $m = i\mu$, with μ real). In all classes Ω is a real number, and represents the uniform angular velocity, or rotation of the material. Without loss of generality, the real constants m in (3) and μ in (5) are both taken as positive.

The prototype of homogeneous universes with rigid rotation is Gödel's cosmos,[4] which in our notation corresponds to the case $m_G^2 = 2\Omega_G^2 > 0$. Gödel's universe permits a remarkable possibility, namely the existence of closed timelike curves (not geodesics!); in other words, it allows for the possibility of an observer to make a round trip and, under proper acceleration, return to the starting point at an instant prior to the moment of departure. Such a violation of causality is made possible essentially because the ordinarily negative metric coefficient $g_{\phi\phi}(r)$ assumes positive values for some range of values of r in the Gödel universe. Indeed, when $m_G^2 = 2\Omega_G^2 > 0$ the coefficient $g_{\phi\phi} = H^2 - D^2$ is negative only inside the cylindrical region enclosing the z axis and having radius r_G given by $\sinh^2(\Omega_G r_G/\sqrt{2}) = 1$. The cylindrical central region $r < r_G \approx 1.25/|\Omega_G|$ is called the causal region of the Gödel universe, while the region laying beyond $r = r_G$ is called noncausal.

An examination of the occurrence of causal and noncausal regions in the homogeneous Gödel-type universes of all three classes given by Eqs. (3)-(5) seems worthwhile.

In the trigonometric class (5) we have $g_{\phi\phi} < 0$ whenever $4\tan^2(\mu r/2) < \mu^2/\Omega^2$. A causal region is then the cylindric central region with radius R given by the minimal positive root of

$$R = (2/\mu)\tan^{-1}(\mu/|2\Omega|) \quad ; \tag{6}$$

such a causal region is surrounded by a noncausal cylindric shell with thickness

$$T = 2\pi/\mu - 2R \quad . \tag{7}$$

Causal and noncausal cylindric shells with thickness 2R and T respectively are next encountered going outwards in an alternate infinite sequence. Novello and Rebouças[5] proposed rotating fluids with anisotropic pressures as possible sources for these universes.

The models of the algebraic class (4) are simpler to describe: now $g_{\phi\phi} = -(1-\Omega^2 r^2)r^2$, so the central cylinder with radius $r=|\Omega|^{-1}$ is causal while the entire outer space is noncausal. A physical source for this spacetime was discovered by Som and Raychaudhuri,[6] who called it "critical".

Finally, the universe models of the hyperbolic class (3) have $g_{\phi\phi} < 0$ whenever $2\tanh(mr/2) < m/|\Omega|$. Since tanh(x) is bounded to 1 for positive x, two subclasses can be distinguished according to whether m is less or greater than $|2\Omega|$. In the first subclass ($m < |2\Omega|$) the situation is similar to that of the preceding algebraic class: there is a central cylindric causal region with radius

$$r_c = (2/m)\tanh^{-1}(m/|2\Omega|) \quad , \; 0 < m^2 < 4\Omega^2 \; , \tag{8}$$

surrounded by the outer noncausal space which extends to the radial infinity; for Ω fixed we remark that the radius (8) of the causal region increases with m, and becomes infinite when $m = |2\Omega|$. In the other subclass ($m > |2\Omega|$) the situation is identical to that of $m = |2\Omega|$: the entire space is causal in the section t=const.

Rebouças and Tiomno[2] investigated the influence of electromagnetic and massless scalar fields on the width of the causal region of hyperbolic universes Eqs. (1),(3); they found that a sourceless electromagnetic field induces a reduction of the ratio $m/|\Omega|$, which implies contraction of the causal region, then going to imaginary m's through m=0. On the contrary, the sourceless scalar field with uniform gradient added to the rotating fluid contributes to enlarge the causal region; in the limit where only the cosmological constant and the scalar field are present, then the condition $m=|2\Omega|$ is at most reached and the noncausal region is exactly excluded, as said before.

Looking for physical realizations of the situation $m^2 > 4\Omega^2$, we investigated the role played by the spin of matter in the disposition of causal/noncausal regions in homogeneous Gödel-type metrics of all three classes Eqs. (3)-(5), and hereby report our findings. To incorporate the spin into a geometry of spacetime we took for granted the Einstein-Cartan-Sciama-Kibble theory,[7] and used Hehl's approach.[7] In this theory a lagrangean is postulated which takes into account the spin properties of matter and fields in a twofold way: i) in the spin dependence of the lagrangean of matter and nongravitational fields themselves; ii) in the form of the connection $\Gamma^{\lambda}_{\mu\nu}$ by imposing the existence of an antisymmetric tensor part $\Gamma^{\lambda}_{[\mu\nu]}$ usually interpreted as a torsion as it appears as a torsion term in the covariant derivatives. The variation of

$\Gamma^{\lambda}_{[\mu\nu]}$ leads to a relation of it with the spin quantities. For a Weyssenhoff fluid with spin density $s_{\mu\nu}$ and four velocity u^{λ} at a point this is $\Gamma^{\lambda}_{[\mu\nu]} = s_{\mu\nu}\, u^{\lambda}$.

2. WEYSSENHOFF FLUID AS A SOURCE OF GÖDEL-TYPE MODELS

In Weyssenhoff[8] fluids the antisymmetric spin density $s_{\mu\nu}$ and the fourvelocity u^{ρ} satisfy $s_{\mu\nu}u^{\nu} = 0$. The dynamics of these fluids was extensively studied by Halbwachs[9], under the special relativistic lagrangian formalism.

In the general relativity, a lagrangian for spinning fluids was proposed by Ray and Smalley[10], who found besides the usual (zero spin) energy-momentum of perfect fluids,

$$\sigma^{Z}_{\mu\nu} = (E+p)\,u_{\mu}u_{\nu} - p\, g_{\mu\nu} \quad ,$$

a metrical contribution linear in the spin density,

$$\sigma^{L}_{\mu\nu} = -(g^{\alpha\beta}+u^{\alpha}u^{\beta})\,(s_{\alpha\mu}u_{\nu}+s_{\alpha\nu}u_{\mu})_{;\beta} \quad .$$

Vaidya et all[11] obtained solutions for fluids with sourceless electromagnetic fields under this elegant, purely metrical approach, while Amorim[12] further enriched the fluid by endowing it with electric charge and magnetic dipole moment.

The Einstein-Cartan[7] theory of gravitation is an attempt to relate the torsion $S_{\mu\nu}{}^{\rho}$ of spacetime to the spin density $s_{\mu\nu}$ of the material content. For Weyssenhoff fluids this relation has been obtained as $S_{\mu\nu}{}^{\rho} = s_{\mu\nu}\, u^{\rho}$ by a large number of authors and in a variety of methods[13-21] (see, however, Refs. 22-25 and the references therein contained for alternative descriptions of spinning fluids).

In the present paper we use the semiclassical model of a spin fluid, as prescribed by Halbwachs[9], Hehl et al[26], Arkuszewski et al[13] and Prasanna[14]. Such a model has been widely used in systems with given spacetime symmetries[13-18], despite an inconvenience it bears[27].

Consider a Weyssenhoff fluid with density ρ, isotropic pressure p, four-velocity u^α and spin density $s_{\alpha\beta}$, satisfying $u^\alpha s_{\alpha\beta}=0$; the symmetric Einstein-Cartan field equations with $8\pi G=c=1$ are[13]

$$G^\mu_\nu = (\rho+p-s^{\alpha\beta}s_{\alpha\beta})u^\mu u_\nu - (p-\tfrac{1}{2}s^{\alpha\beta}s_{\alpha\beta})\delta^\mu_\nu + (g^{\alpha\beta}+u^\alpha u^\beta)\nabla_\alpha(u^\mu s_{\nu\beta}+u_\nu s^\mu{}_\beta) \quad , \tag{9}$$

where G^μ_ν is the riemannian Einstein tensor and ∇_α is the riemannian covariant derivative. Also from Ref. 13, the antisymmetric equations for the Weyssenhoff fluid may be written as

$$u^\alpha u^\beta \nabla_\alpha(u_\mu s_{\nu\beta} - u_\nu s_{\mu\beta}) + \nabla_\alpha(s_{\mu\nu}u^\alpha) = 0 \quad . \tag{10}$$

In the same spirit as above, Eq. (10) is the equation of motion for the spin density $s_{\mu\nu}$ in presence of the gravitational field. Forgetting the interpretation of the manifold as a spacetime with metric $g_{\mu\nu}$ and torsion $\Gamma^\alpha_{[\mu\nu]}$ which lead to Eqs. (9), (10), we may consider Eq. (9) as the Einstein equation for a riemannian space with metric $g_{\mu\nu}$ generated by a source formed by a fluid endowed with spin which, besides being acted upon by the gravitational field $g_{\mu\nu}$ (Eq. (10)), is itself a source of $g_{\mu\nu}$ (Eq. (9)). We shall analyse the properties of spacetime homogeneous Gödel-type solutions of these equations in the light of the Raychaudhuri--Thakurta, Rebouças-Tiomno classification.[1,2]

We assume the fluid to be at rest in the standard reference system of Eqs. (1), (3), so we set $u^\alpha = \delta^\alpha_0$; we also assume that the spin is parallel to the z axis (only the component $s_{r\phi}$ survives) and that it is homogeneous ($s_{r\phi}s^{r\phi}$ is constant), so we set

$$s_{r\phi}(r) = s\, D(r) \quad , \quad s = \text{const.} \tag{11}$$

With these assumptions all terms in Eq. (10) are zero, while the independent field equations in (9) are

$$G^t_t \equiv 3\Omega^2 - m^2 = \rho - s^2 - 4s\Omega \ , \tag{12}$$

$$G^r_r \equiv -\Omega^2 = -p + s^2 + 2s\Omega \ , \tag{13}$$

$$G^z_z \equiv \Omega^2 - m^2 = -p + s^2 \ , \tag{14}$$

where the terms in $s\Omega$ arose from $\nabla_\alpha(u_{(\mu}s_{\nu)}{}^\alpha)$ in Eq. (9).

As already shown in Ref. [3], the equations (12)-(14) are still valid for inhomogeneous Gödel-type spaces; then the quantities s, m and Ω depend on the radius r, submitted to one constraint $s+\Omega=$const.

Returning to the present case of homogeneous spaces, we have five constants (ρ, p, s, m, Ω) satisfying three relations only; as only two parameters are independent, we write the solution as

$$\rho = p = (\Omega+s)^2 \quad , \quad m^2 = 2\Omega(\Omega+s) \quad . \tag{15}$$

One immediately sees from (15) that the density ρ is non-negative, and that we are dealing with stiff matter ($\rho=p$), as for the Gödel universe with pressure and the cosmological constant $\Lambda = 0$.[2] One also sees that all positive and negative values for m^2 are possible ($-\infty < m^2 < +\infty$), the sign of m^2 depending solely on the value of the ratio s/Ω (thus $m^2 \lessgtr 0$ if $s/\Omega \lessgtr -1$).

3. EFFECT OF THE INTRODUCTION OF SPIN WITH FIXED $\Omega + s$

To get a clear picture of the influence of the spin upon the metrics we shall compare our system containing spin with a similar, but spinless system, namely Gödel's cosmos. In the spirit of the present paper, the ingredient of Gödel's universe is taken as spinless stiff matter with uniform rotation Ω_G, satisfying

$$\rho_G = p_G = \Omega_G^2 \quad , \quad (m_G^2 = 2\Omega_G^2) \quad . \tag{16}$$

In a gedanken operation we now impress spin s into the matter of the Gödel universe, with the proviso that the density ρ of the fluid is kept unaltered in the process ($\rho=\rho_G$). From (15), we see that necessarily also the pressure p and the combination $\Omega + s = \Omega_G$ are held constant in the process. The metric parameters Ω and $m^2 = -\mu^2$ then depend on the fixed value of Ω_G and the variable value of the spin s according to

$$\Omega = \Omega_G - s \quad , \quad m^2 = -\mu^2 = 2\Omega_G(\Omega_G - s) \quad , \; \Omega_G = \text{const} \quad . \tag{17}$$

For future reference we rewrite the critical radius (8) now in terms of fixed Ω_G and variable s, for $0<m^2<4\Omega^2$,

$$r_c = |\sqrt{2}/\Omega_G|(1-s/\Omega_G)^{-1/2}\coth^{-1}(2-2s/\Omega_G)^{1/2} \quad , \tag{18}$$

while the widths 2R and T of the causal and noncausal regions Eqs. (6) and (7) are written, now for $m^2 = -\mu^2 < 0$,

$$2R = |2\sqrt{2}/\Omega_G|(-1+s/\Omega_G)^{-1/2}\cot^{-1}(-2+2s/\Omega_G)^{1/2} \quad , \tag{19}$$

$$T = |2\sqrt{2}/\Omega_G|(-1+s/\Omega_G)^{-1/2}\tan^{-1}(-2+2s/\Omega_G)^{1/2} \quad , \tag{20}$$

where the minimal positive values of the functions $\cot^{-1}$ and $\tan^{-1}$ are to be taken.

The plan of presentation of our results is as follows: we take Gödel's universe and insert spin s initially with the same sign as the Gödel rotation Ω_G, and describe the evolution of the system while the ratio s/Ω_G increases from the value zero to $+\infty$. Next we start again from Gödel's solution and introduce spin, but now with the opposite sign; we describe the system while the negative ratio s/Ω_G then changes from zero to $-\infty$.

Case $s/\Omega_G > 0$

Gödel's universe has a central, causal region with radius $r_G = 1.25|\Omega_G|^{-1}$. Upon insertion of small spin s with $s/\Omega_G > 0$ the rotation $|\Omega|$ diminishes, $m^2 > 0$ decreases (see Eq. (17)), and the radius r_c of the central causal region increases (see Eq.(18)).

For increasing $s/\Omega_G > 0$ the region of causality is enlarged more and more, until it becomes infinite when $s/\Omega_G=1/2$. At this stage the rotation Ω has been reduced to $\Omega_G/2$, the half of the initial value, being equal to s. Also $\rho = p = \Omega_G^2$. This is the geometrical system ($m^2 = 4\Omega^2$) first found by Rebouças and Tiomno[2] as generated by a scalar field in empty space, with cosmological constant $\Lambda \neq 0$. Here, however, there is a material background with a non-null vorticity, while in that case it is not simple to justify that the vorticity calculated from the field of velocities $u^{\alpha} = \delta^{\alpha}_{0}$ represents a real rotation of the universe.

We proceed increasing the ratio $s/\Omega_G > 1/2$; the rotation $|\Omega|$ continues to decrease, and the space remains causal in all its radial extent.

When $s/\Omega_G = 1$ the rotation Ω is zero, and we have $\rho = p = s^2$. As already remarked by Arkuszewski et al.,[13] the combinations $\rho_{eff} = \rho - s^2$ and $p_{eff} = p - s^2$ behave as effective energy density and effective pressure, respectively. Universes of this type, with the ingredients ρ, p, s alone, were studied by Prasanna;[14] when both ρ_{eff} and p_{eff} vanish, the spacetime is Riemann-Christoffel-flat. In a sense, Einstein-Cartan models obtained from Einstein solutions solely with this correspondence are trivial; the models considered in this paper are nontrivial due to the $s\Omega$ terms in Eqs. (12) and (13). To our knowledge, these are the only nontrivial solutions found so far.

We next increase s/Ω_G to a value a little larger than 1; the rotation Ω is small, and its sign is now opposite to that of s and Ω_G. This is a region of transition from the m^2 to the μ^2 case: from (17) we see that m^2 is negative now, so we prefer to use $\mu^2 = 2\Omega_G(s-\Omega_G)$ and the corresponding trigonometric expressions (5). The central causal region is a cylinder with a large radius, $R \simeq \pi/[2\Omega_G(s-\Omega_G)]^{1/2}$. The cylinder is surrounded by a much thinner noncausal region, with thickness $T \simeq 4|\Omega_G|^{-1}$; going outwards, successive causal regions with large thickness $2R$ alternating with noncausal regions with thickness T are encountered.

With increasing values of $s/\Omega_G > 1$ the values of R and T both decrease, but R diminishes faster than T; in other words, the predominance of the causal regions over the noncausal ones diminishes. The rotation $|\Omega|$ is increasing, with Ω having sign opposite to that of the spin s. When $s/\Omega_G = 3/2$ the widths 2R and T become equal $(=\pi/|\Omega_G|)$, while for $s/\Omega_G > 3/2$ the noncausal regions predominate $(T > 2R)$. As always, the central region is causal.

Finally, for large positive values of s/Ω_G the rotation $|\Omega|$ also becomes large, and Ω has sign opposite to that of Ω_G and of large s. The noncausal regions are now thin, $T \doteq \sqrt{2}\pi/(s\Omega_G)^{1/2}$, but the causal regions are much thinner, $R \doteq |s|^{-1}$. This completes the description of systems with positive ratio s/Ω_G.

Case $s/\Omega_G < 0$

In the same spirit we now return to the spinless Gödel universe with rotation Ω_G, and introduce spin s now with sign opposite to that of Ω_G. From (17) one sees that m^2 is positive for all values of the negative ratio s/Ω_G, so the hyperbolic expressions (3) for the metric coefficients are used; in all circumstances we have a central causal region with radius r_c given by (18), while the outer space is noncausal and extends to the radial infinity.

When $s/\Omega_G=0$ we have Gödel's universe, with critical radius $\doteq 1.25|\Omega_G|^{-1}$; for Ω_G fixed and ever decreasing values of the negative ratio s/Ω_G the radius r_c of the single causal region shrinks monotonically. For large values of negative s/Ω_G it behaves as $|\Omega|^{-1} \doteq |s|^{-1}$, thus coinciding with the radius R of the thin central causal region encountered when $s/\Omega_G \to +\infty$. Such a coincidence is not accidental: with Ω_G fixed, the geometries of the two limiting cases $s/\Omega_G \to \pm\infty$ both tend to one and the same spacetime, the "critical" solution mentioned by Som and Raychaudhuri.[6] Here, however, this solution is produced in absence of electromagnetic field and the fluid has spin.

4. EFFECT OF THE INTRODUCTION OF SPIN WITH Ω FIXED

The remark made at the end of the preceding Section is made clearer if we study the evolution of the spacetime as we insert spin s into Gödel's universe while keeping invariant the rotation Ω (instead of keeping unaltered the combination $\Omega + s = \Omega_G$ as we have done so far): with $\Omega = \text{const}$ we rewrite (15) preferably as

$$\rho = p = \Omega^2 (1+s/\Omega)^2 \ , \quad m^2 = 2\Omega^2 (1+s/\Omega) = -\mu^2 \ , \qquad (21)$$

and now study the variation of $\rho = p$ and of $m^2 = -\mu^2$ as functions of s/Ω.

Case $s/\Omega < 0$

We start again from Gödel's universe, for which $s=0$, $\rho = p = \Omega^2$, $m^2 = 2\Omega^2$, $r_c = 1.25|\Omega|^{-1}$, and insert spin s initially with sign opposite to that of the rotation Ω ; in the process, the value of the rotation Ω will be kept invariant. From Eq. (21), for decreasing values of the negative ratio $-1<s/\Omega < 0$, the density and pressure ($\rho=p$) diminish; also the value of positive m^2 diminishes. With Ω fixed and positive m^2 decreasing, Eq. (8) says that the central causal region shrinks along this phase of the process; the region is surrounded by an infinite noncausal space.

From Eq. (21), when s/Ω reaches the value -1 then m^2 vanishes, as in Som-Raychaudhuri[6] solution; the radius of the single causal region is reduced to $|\Omega|^{-1}$ (see Eq. (8)), and the noncausal space still extends to the radial infinity. However, the density and pressure now vanish, so the spin alone would be responsible for the geometry of spacetime. The possibility of a universe endowed with spin solely is somewhat disturbing; it has been proposed a spinless, electrically charged material as a possible source for this "critical" spacetime.[6]

We now arrive to the point we wanted to discuss: the abrupt transition from the case $s/\Omega_G = -\infty$ to the case $s/\Omega_G = +\infty$ for Ω_G finite and fixed. In the present sequence of situations with Ω fixed, it corresponds not to an abrupt, but to a smooth transition from the case $s/\Omega = -1+\epsilon$ into the case $s/\Omega = -1-\epsilon$ (with ϵ small and positive). This is a region of transition from the m^2 case to the μ^2 case. When $s/\Omega = -1-\epsilon$ the density and pressure in Eq. (21) are small, $\rho = p = \epsilon^2\Omega^2$, and the central causal region has radius $\simeq (1-\epsilon/6)|\Omega|^{-1}$ as seen from Eq. (6) with $\mu^2 = 2\epsilon/\Omega^2$. However, the previously infinite noncausal space now has a finite, though large thickness (Eq. (7)): $T = \pi(2/\epsilon)^{1/2}|\Omega|^{-1}$. Beyond the noncausal region, new causal regions $\simeq 2|\Omega|^{-1}$ thick and noncausal regions $\simeq 4.4/|\sqrt{\epsilon}\Omega|$ thick now develop in alternation.

For completeness, we also describe the evolution of the spacetime while s/Ω decreases from -1 to $-\infty$. With the rotation Ω fixed and the ratio s/Ω decreasing from -1 to $-\infty$, the density and pressure (21) both increase from zero to infinity. The value of $\mu^2 = -m^2$ also increases from zero to infinity. The thickness $2R$ of the causal regions (6) and T of the noncausal regions (7) both decrease, and both tend to zero when $s/\Omega \to -\infty$. However, T decreases as fast as $|s/\Omega|^{-1}$, while 2R decreases simply as $|s/\Omega|^{-1/2}$; as a consequence, the entire space tends to be causal in the limit $s/\Omega \to -\infty$.

Case $s/\Omega > 0$

With the rotation Ω finite and fixed, the transition from the case $s/\Omega = -\infty$ to the case $s/\Omega = +\infty$ again seems abrupt, since it demands that the spin s changes from an infinite value to the infinity of the opposite sign. However, the cases $s/\Omega = \pm\infty$ correspond to one and the same spacetime, if one assumes that Ω vanishes instead of that s diverges. For vanishing Ω we have ρ, p and s as sole physical variables, a problem studied by Prasanna;[14] the universe with $\Omega = 0$, $\rho = p = s^2$ is flat, in the sense that the corresponding Riemann-Christoffel tensor is null. This is easily seen from Eq. (15): m^2 vanishes when s is finite and $\Omega = 0$; with m and Ω both null, the line element (1), (4) is Minkowskian in cylindrical coordinates. Actually the results is trivial , as $\rho_{eff} = p_{eff} = \Omega = 0$; thus this universe is "effectively" empty.

We now describe the system while the ratio s/Ω decreases from $+\infty$ to 1, with the rotation Ω finite and fixed. The density and pressure (15) decrease from ∞ to $\rho = p = 4\Omega^2$; the parameter m^2 (15) is positive and $\geq 4\Omega^2$, so the entire space is causal (see Eq. (8)).

The transition from the case $s/\Omega = 1$ to the case $s/\Omega = 1-\epsilon$ (with ϵ positive and small) is interesting: a noncausal region developes from the radial infinity until $r_c = |2\Omega|^{-1} \ell n\, \epsilon^{-1}$. The density and pressure are continuous in the transition.

Finally, when s/Ω changes from 1 to zero, with the rotation Ω finite and fixed, the quantities $\rho = p$ decrease monotonically from

$4\Omega^2$ to Ω^2. The value of m^2 is positive, but is less than $4\Omega^2$ and is decreasing; as a consequence, the radius r_c of the central causal cylinder shrinks to $r_G \simeq 1.25|\Omega|^{-1}$, the critical radius of Gödel's universe. We have thus returned to the original spinless situation (s=0).

5. FINAL REMARKS

The present paper is a specialization of a previous work of ours[3], valid for inhomogeneous spaces; such a specialization, with restriction to homogeneous Gödel-type spaces, makes easier the analysis of the gravitational peculiarities of the intrinsic spin.

In a recent paper, Bedran et al[28] considered a metric similar to (1), but with $g_{rr} = g_{zz} = -G(r)$ instead of -1; their spacetime is thus not of the Gödel type. In the limiting case of vanishing pressure and spin they recover van Stockum's[29] solution, while setting G=1 the system becomes of the Gödel type and the equations of Ref. [3] are reobtained.

Still more recently, Smalley[23] applied the self consistent formulation[19] of the spin fluid to the Gödel cosmology. Differently from us and from Bedran et al[28], however, he inserted the spin without changing the metric. In the process, he found that the cosmological constant became slightly more negative, and that the magnitude of the angular velocity remained unaltered but the sense of rotation was flipped 180^0.

REFERENCES

[1]A.K. Raychaudhuri and S.N. Guha Thakurta, Phys. Rev. D 22, 802-806 (1980).

[2]M.J. Rebouças and J. Tiomno, Phys. Rev. D 28, 1251-1264 (1983) and references therein quoted.

[3]J. Duarte de Oliveira, A.F. da F. Teixeira and J. Tiomno, Contributed Paper to the 10^{th} International Conference on General Relativity and Gravitation, Padova (Italy) 1983.

[4]K. Gödel, Rev. Mod. Phys. 21, 447-450 (1949).

[5]M. Novello and M.J. Rebouças, Phys. Rev. D 19, 2850-2852 (1979); in their paper the central region is noncausal due to different choice of integration constants, which makes their coordinates noncylindrical.

[6]M.M. Som and A.K. Raychaudhuri, Proc. Roy. Soc. London A 304, 81-86 (1968).

[7]F.W. Hehl, P. von der Heyde, G.D. Kerlick and J.M. Nester, Rev. Mod. Phys. 48, 393-416 (1976) and references therein; also F.W. Hehl, Gen. Rel. Grav. 5, 491-516 (1974).

[8]J. Weyssenhoff and A. Raabe, Acta Phys. Polonica 9 (1974) 7-18.

[9]F. Halbwachs, "Théorie relativiste des fluids à spin" (Gauthier-Villars, Paris (1960).

[10]J.R. Ray and L.L. Smalley, Phys. Rev. D26 (1982) 2619-2622.

[11]E.P.V. Vaidya, M.L. Bedran and M.M. Som, Progr. Theor. Phys. 72 (1984) 857-859.

[12]R. Amorim, Phys. Letters 104 A (1984) 259-261.

[13]W. Arkuszewski, W. Kopczyński and V.N. Ponomariev, Ann. Inst. Henri Poincaré 21 A (1974) 89-95; they call s/2 what we call s.

[14]A.R. Prasanna, Phys. Rev. D11 (1975) 2076-2082 and 2083-2087.

[15]M.M. Som and M.L. Bedran, Phys. Rev. B24 (1981) 2561-2563; J. Phys. A15 (1982) 3539-3542.

[16]M.M. Som, M.L. Bedran and C.M. Amaral, Progr. Theor. Phys. 67 (1982) 683-688.

[17]D. Tsoubelis, Phys. Rev. Letters 25 (1983) 2235-2237; Phys. Rev. D30 (1984) 1632-1636.

[18]W. Kopczyński, Phys. Letters 43A (1973) 63-64; M.L. Bedran and L.C.C. Andrade, Progr. Theor. Phys. 70 (1983) 1583-1587.

[19]J.R. Ray and L.L. Smalley, Phys. Rev. Letters 49 (1982) 1059-1061; E50 (1983) 623; Phys. Rev. D27 (1983) 1383-1385.

[20]R. de Ritis, M. Lavorgna, G. Platania and C. Stornaiolo, Phys. Rev. D28 (1983) 713-717; Phys. Letters 95A (1983) 425-428; for charged spin fluid, see Phys. Rev. D31 (1985) 1854-1859.

[21]R. Amorim, Phys. Rev. D31 (1985) 3099-3103.

[22]G. Cognola, R. Soldati, L. Vanzo and S. Zerbini, Phys. Rev. D25 (1982) 3109-3116; Nuovo Cimento 76B (1983) 109-129.

[23]L.L. Smalley, Phys. Rev. D32 (1985) 3124-3127; Phys. Letters 113A (1986) 463-466.

[24]R. Amorim, Phys. Rev. D33 (1986) 2796-2800.

[25]J.L. Ray and L.L. Smalley, Gen. Relat. Gravit. 18 (1986) 549-553.

[26]F.W. Hehl, P. von der Heyde and G.D. Kerlick, Phys. Rev. D10 (1974) 1066-1069.

[27]R. Amorim and J. Tiomno, Rev. Bras. Fis. 8 (1978) 350-367 and references therein.

[28]M.L. Bedran, E.P.V. Vaidya and M.M. Som, Nuovo Cimento 87B (1985) 101-107.

[29]W.J. van Stockum, Proc. R. Soc. Edinburgh 57 (1937) 135-154.