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ON THE MEASURE OF FRACTAL DIMENSIONALITIES THROUGH PHYSICAL PROPERTIES

by

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ABSTRACT

The measure of the *fractal dimensionalities* of real substances through direct use of the mathematical definitions can be extremely cumbersome. We claim that in the case of arbitrarily rough and irregular conductors, the task can be considerably simplified by using the traditional skin effect. Indeed both *electrical resistance* and *electromagnetic power dissipation* should present anomalous power-law dependences on the applied frequency, the corresponding exponents being directly related to relevant fractal dimensionalities.

1 INTRODUCTION

An enormous amount of real physical systems present, within appropriate length ranges, a fractal nature which is commonly characterized by the values of one or more relevant fractal dimensionalities. The usefulness of such approach in the case of porous coals[1], proteins[2,3], sintered metallic powders[4], cement gels[5] and many other substances[6] needs no more to be proved. However the practical determination of the fractal dimensionalities is quite frequently cumbersome, which in turn implies in results whose degree of confidence is not always satisfactory. We claim[7] that the traditional skin effect can be an excellent tool for overcoming the above difficulties whenever we are dealing with substances presenting a non neglectable electrical conductivity.

There are physical properties (e.g., spin relaxation[2], specific heat[8]) which are closely related to both *static* (purely geometric) and *dynamic* fractal *dimensionalities*. For example, the low temperature (T) spin-lattice relaxation time t_1 of some hemoproteins is given[2] by $1/t_1 \propto T^n$, where the non-integer exponent n can be related to a non-integer dimensionality $\bar{d}$ through[2,3] $n=3+2\bar{d}$. Under certain circumstances, the dimensionality $\bar{d}$ can be identified[3] with the *fracton dimensionality*[9] $d_{fr}=2d/d_w$, where d_f is the structural fractal dimensionality of the hemoprotein, and d_w is the fractal dimensionality of a random walk (not self-avoiding) constrained to the fractal (d_w is defined through $\langle r^2\rangle^{d_w} \sim t^2$, where $\langle r^2\rangle$ is the mean square end-to-end distance of the walk after t steps). We then see that the experimental data (say n) yields information on the purely geometric dimensionality d_f, but the relationship is rather indirect, and mixed with the "dynamic" dimensionality d_w (reflecting diffusion and vibrational aspects of the problem). Consequently we would hardly consider spin relaxation measurements as a practical tool for determining d_f.

On the other hand, other physical phenomena (e.g., small-angle X-ray scattering[1,6]) exist which provide experimental data *directly* related to *purely geometric* fractal dimensionalities of the system, and are therefore convenient for determining those quantities. For example, in the just quoted X-ray experiments, the scattering intensity I is proportional[1] to q^{d_s-6} where q is the scattering angle (radians). Consequently a $\ln I$ vs. $\ln q$ representation of the experimental data immediately provides the fractal surface dimensionality d_s (which equals 2 in the Euclidean case, thus providing the well known $I \propto 1/q^{-4}$ law). We argue herein that the standard skin effect belongs to this same category of methods, and should therefore constitute a convenient way for determining fractal dimensionalities of arbitrarily rough and irregular conductors, the use of the mathematical definitions of those quantities being replaced by relatively simple physical measurements.

2 SKIN EFFECT

Let us consider a roughly cylinder-like conductor (see Fig. 1). We denote ℓ_L and ℓ_T its fractal longitudinal and transverse (perimeter) lengths respectively, and S the fractal area they support. For arbitrary fractal surfaces it will in general be $S \neq \ell_L \ell_T$. We assume that an alternate electric potential with voltage V and frequency ω is applied along the cylinder. The corresponding skin depth is given[10] by

$$\delta \propto \frac{1}{\sqrt{\omega}}$$

where the proportionality factor depends on the electromagnetic properties (such as the electrical conductivity) of the (homogeneous) substance.

The fractal quantities ℓ_L, ℓ_T and S respectively yield the fractal dimensionalities d_L, d_T and d_S through the following relations:

$$d_L = \frac{\ln(\ell_L/\delta)}{\ln(1/\delta)} \qquad (2)$$

hence

$$\ell_L = \frac{1}{\delta^{d_s - 1}} , \qquad (2')$$

$$d_T = \frac{\ln(\ell_T/\delta)}{\ln(1/\delta)} \qquad (3)$$

hence

$$\ell_T = \frac{1}{\delta^{d_T - 1}} , \qquad (3')$$

and

$$d_S = \frac{\ln(S/\delta^2)}{\ln(1/\delta)} \qquad (4)$$

hence

$$S = \frac{1}{\delta^{d_S - 2}} \tag{4'}$$

It will in general be $1 \leq d_L$, $d_T < 2$ (the equality holds for smooth differentiable curves), and $2 \leq d_S < 3$ (the equality holds for smooth differentiable surfaces). An interesting particular situation is that in which $S \simeq \ell_L \ell_T$, which implies $d_S = d_L + d_T \geq 2$. Another interesting particular situation is that of a (statistically) homogeneous and isotropic fractal surface; in that case it will in general be $d_L = d_T = d_S - 1 \geq 1$.

The power P dissipated by the substance is given by

$$P = \int \vec{j}(\vec{r}).\vec{E}(\vec{r})d^3r \tag{5}$$

where $\vec{j}(\vec{r})$ and $\vec{E}(\vec{r})$ respectively are the current density and the electric field at the point $\vec{r}$. By using Ohm's law $\vec{j}(\vec{r}) = \sigma \quad \vec{E}(\vec{r})$ ($\sigma \equiv$ electrical conductivity), Eq. (5) becomes

$$\begin{aligned} P &= \sigma \int E^2 d^3r \\ &= \sigma \langle E^2 \rangle_{skin} \int_{skin} d^3r \\ &= \sigma \langle E^2 \rangle_{skin} \; S\delta \end{aligned} \tag{6}$$

where "skin" refers to the volume $V \equiv \int_{skin} d^3r \simeq S\delta$ of the substance where the electric field is sensibly different from zero (we recall that the electric field vanishes exponentially while entering into the material).

If we perform a *fixed electromagnetic field density experiment* (in the interior of an appropriate cavity), Eq. (6) yields

$$P \propto S\,\delta \simeq \delta^{3-d_S} \tag{7}$$

where we have used Eq. (4'). Therefore, using Eq. (1), we obtain

$$P \propto \frac{1}{\omega^{\frac{3-d_S}{2}}} \tag{8}$$

This relation reproduces the standard one[10] ($P \propto 1/\sqrt{\omega}$) when $d_S = 2$. Note also that Eq. (7) implies $v = \delta^{3-d_S}$, which through appropriate change of the characteristic length, recovers Eq. (4) of Ref. 1.

If we perform instead an *electric current flow experiment* we will have $\langle E^2 \rangle_{skin} \simeq V^2/\ell_L^2$ hence, using Eq. (6),

$$P = \frac{\sigma V^2 S \delta}{\ell_L^2} \tag{9}$$

which, if identified with $P = V^2/R$ (R being the electrical resistance), provides

$$R = \frac{\sigma \ell_L^2}{S\delta} \tag{10}$$

Finally, using Eqs. (2') and (4'), we obtain

$$R \propto \delta^{d_S - 2d_L - 1} \tag{11}$$

hence, through Eq. (1),

$$R \propto \omega^{\frac{1+2d_L-d_S}{2}} \tag{12}$$

This relation reproduces the standard one[10] ($R \propto \sqrt{\omega}$) when $d_S=2$ and $d_L=1$.

Summarizing, the representation of the experimental data in $\ln P$ vs. $\ln\omega$ and $\ln R$ vs. $\ln\omega$ graphs would provide straight lines whose respective slopes would be $(d_S-3)/2$ and $(1+2d_L-d_S)/2$. The determination of d_S and d_L would then be straighforward.

In the particular case mentioned before, namely $d_S=d_L+d_T$, Eqs. (8) and (12) provide

$$P \propto \frac{1}{\omega^{\frac{3-d_L-d_T}{2}}} \tag{13}$$

and

$$R \propto \omega^{\frac{1+d_L-d_T}{2}} \tag{14}$$

In the other particular case (quite frequent in nature), namely $d_L=d_T=d_S-1$, Eqs. (8) and (12) provide

$$P \propto \frac{1}{\omega^{\frac{2-d_L}{2}}} \tag{15}$$

and

$$R \propto \omega^{d_L/2} \tag{16}$$

In this case a single experience, say R vs. ω, would determine d_L.

In both types of experiment (electric current flow and electromagnetic cavity) the spatial distribution of the electromagnetic field presents an evolution with ω: it is through this evolution that the field probes the fractality of the system. We have qualitatively represented in Fig. 2.a (Fig. 2.b) the spatial distribution of the electric field outside (inside) of the substance in a electromagnetic cavity (electric current flow) experiment.

3 CONCLUSION

As a consequence of the scaling arguments presented above, the skin effect appears to be a promising tool for measuring fractal dimensionalities of electrically conducting materials. Real substances are naturally not expected to exhibit this type of anomalous skin effect for all frequencies. Therefore an actual experiment, say a resistance measurement, should present various regimes, making crossovers from one into the other at appropriate frequencies. A *low frequency regime* will always be present in which the resistance is independent from ω (δ larger than the transverse linear size of the sample); this regime ends when δ becomes comparable with this size. A *high frequency regime* will also exist always: it corresponds to the break-down of the condition $\omega \ll \sigma/\varepsilon$ ($\varepsilon \equiv$ dielectric constant), which is necessary for the validity of Eq.1 (if $\omega > \sigma/\varepsilon$, the polarizability itself starts depending on ω, and non trivial effects appear). Between these low and high frequency regimes, one or more fractal and/or euclidean skin regimes can appear, corresponding to scales of δ within which the surface of the conductor is seen as "rough" or "smooth". In Fig. 3 we have illustrated these concepts by assuming, in the intermediate, frequency region, a fractal regime (slope $(1+2d_L-d_S)/2$ different from 1/2) followed by a standard one (slope 1/2).

Let us conclude by saying that experiments testing Eqs. (8) and (12) would be extremely welcome. Systems like copper, gold, silver porals[4], porous coals, metallic sponges could be good candidates.

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FIGURE CAPTIONS

Fig. 1 - Cylinder-like conducting sample with fractal external surface; ℓ_T refers to the perimeter. The alternate voltage is applied between the two (equipotential) bases. The indicated skin depth δ is out if scale (too large) if assumed to be appropriate for probing the "fractality" of the illustrated "rough" surface.

Fig. 2 - Magnified view of the rough surface of the conductor: the dashed line qualitatively indicates the skin of the substance at a particular frequency. For the cavity experiment we have indicated ((a) and (b)) the electric field *outside* of the conductor; for the current flow experiment we have indicated ((c) and (d)) electric field *inside* of it.

Fig. 3 - Possible result for a resistance measurement of a conducting rough sample, exhibiting a fractal regime (slope $\neq$ 1/2) which crosses over to a standard one (slope 1/2). $\omega_0(R_0)$ is a reference frequency (resistance) adapted to the particular sample.

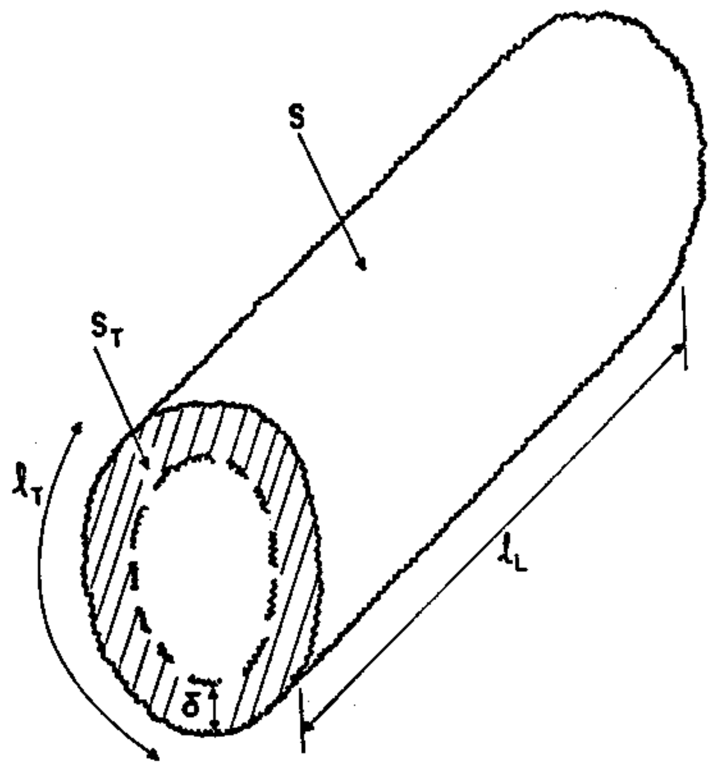

FIG. 1

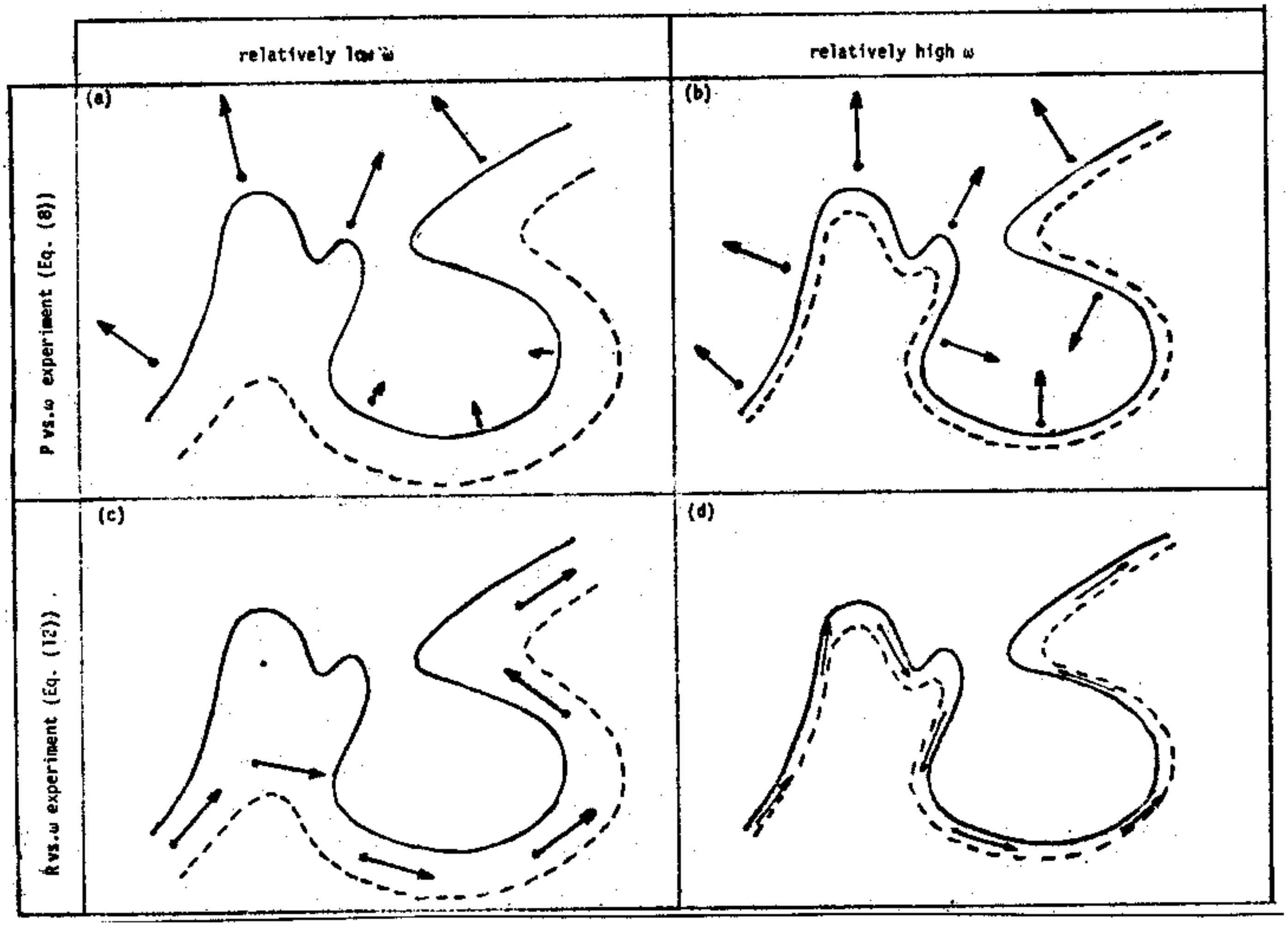
relatively low ω
relatively high ω
(a)
(b)
(c)
(d)
P vs.ω experiment (Eq. (8))
k vs.ω experiment (Eq. (12))

FIG. 2

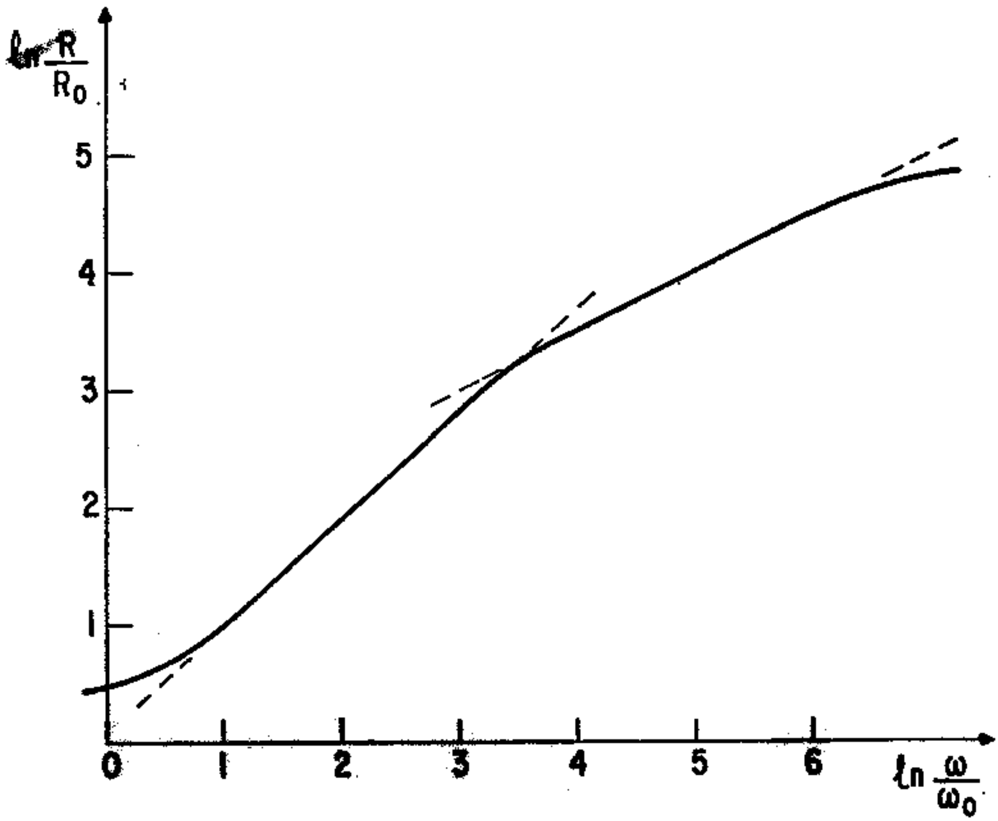

FIG. 3

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